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SOLUTION TO PROBLEMS OF SCHMIDT AND QUACKENBUSH FROM 1974 AND 1985: TENSOR PRODUCTS OF SEMILATTICES

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Abstract: If M is a finite complemented modular lattice with n atoms and D is a bounded distributive lattice, then the Priestley power $M[D]$ is shown to be isomorphic to the poset of normal elements of D^n, thus solving a problem of E. T. Schmidt from 1974. It is shown that there exist a finite modular lattice A not having M_4 as a sublattice and a finite modular lattice B such that $A \otimes B$ is not semimodular, thus refuting a conjecture of Quackenbush from 1985. It is shown that the tensor product of M_3 with a finite modular lattice B is supersolvable if and only if B is distributive, thus proving a conjecture of Quackenbush from 1985.

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1. Introduction, motivation, and context

In this note we resolve three conjectures that directly or indirectly relate to tensor products of semilattices. All three can be resolved essentially by the same device. Although the conjectures concern finite lattices, in order to best understand this device it is preferable to consider lattices without regard to cardinality. In this section, we introduce the ideas that unite the three conjectures and the related background. This section is primarily intended for readers familiar with Priestley duality, but for readers without this background we also present definitions and examples. For notation or definitions not defined here, see §2, [3], or [5].

The first conjecture pertains to function lattices, so we define these first, along with two important topologies. These concepts are included so that a general reader will have some idea of where the results sit within the broader framework of the field, and so that the main results of this paper are not viewed merely as disconnected statements.

Let L and P be posets. (Generally, L is a lattice or semilattice.) Let L^P denote the poset of order-preserving maps from P to L, where as usual we order the functions pointwise:

$$f \leqslant g \text{ if } f(p) \leqslant g(p) \text{ for all } p \in P.$$

Example. Let $P = \mathbf{2} = \{0, 1\}$ and let $L = \mathbf{3} = \{0, 1, 2\}$. Then $\mathbf{3}^{\mathbf{2}}$ is the set of pairs (a, b) where $a, b \in \mathbf{3}$ and $a \leqslant b$:

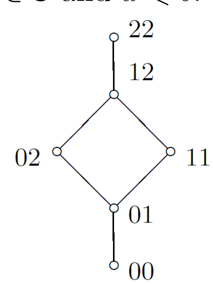

Figure 1.1. The poset $\mathbf{3}^{\mathbf{2}}$

If P is an antichain, then L^P is just the usual direct product of $|P|$ copies of L.

By L^σ, on the other hand, we mean the poset of *ideals* of L. A non-empty subset $I \subseteq L$ is an *ideal* if

(1) I is a down-set;
(2) $a \vee b \in I$ for all $a, b \in I$.

(If L is a Boolean algebra, then under the isomorphism between the categories of Boolean algebras and Boolean rings, I is a lattice ideal if and only if I is a ring ideal [15, §11].) For this definition, we need L to at least be a $\vee$-semilattice; for more general posets, an *ideal* is a directed down-set. See the diagram of a $\vee$-semilattice S and its ideal lattice in Fig. 1.2, consisting of the chain $\mathbb{N}$ and three elements x, y, z greater than each natural number such that $x, z < y$.

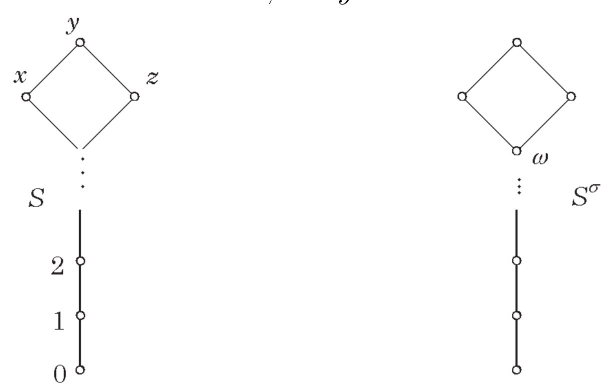

Figure 1.2. The $\vee$-semilattice S (which is not a lattice) and its ideal lattice S^σ

If P and Q are partially ordered topological spaces, let Q^P_τ denote the set of all *continuous* order-preserving functions from P to Q, where Q has topology τ. (If τ is absent, we assume Q has the discrete topology.)

It is well known that every member L of the category $\mathbf{D}$ of bounded distributive lattices is isomorphic to $\mathbf{2}^P$, where P belongs to the category of *Priestley spaces*, compact, totally order-disconnected ordered spaces + continuous order-preserving maps. (A partially ordered topological space P is *totally order-disconnected* if, whenever $p \nleqslant q$ in P, there exists a clopen up-set U such that $p \in U$ but $q \notin U$.) Thus $L \cong D(P)$, the lattice of clopen up-sets of P, and $P \cong P(L)$, the poset of prime filters of L with the topology generated by the following subbasis [18]:

$$\big\{\{F \in P(L) \mid a \in F\}, \{F \in P(L) \mid a \notin F\} \bigm| a \in L\big\}.$$

Recall that an element k in a complete lattice A is *compact* if, for all $T \subseteq A$ such that $k \leqslant \bigvee T$, there exists a finite subset $S \subseteq T$ such that $k \leqslant \bigvee S$. The complete lattice A is *algebraic* if every element is a join of compact elements. (Examples include the lattice of subgroups of a group, the compact subgroups being the finitely-generated ones, and the lattice of open sets of a topological space for which the compact opens form a basis.) Every algebraic lattice A can be represented as S^σ for some $\vee$-semilattice S with least element 0. (Take S to be the set $\kappa(A)$ of compact elements of A.) Two interesting topologies are the *Scott topology* Σ, generated by the basis

$$\{\uparrow k \mid k \in \kappa(A)\}$$

and the *Lawson topology* Λ, generated by the subbasis

$$\Sigma \cup \{A \backslash \uparrow k \mid k \in \kappa(A)\}.$$

Every algebraic lattice with the Lawson topology is a Priestley space. (Algebraic lattices are the simplest examples of the *continuous lattices* invented by Dana Scott at Oxford and used in the semantics of programming languages [10].)

Note that if $L, M \in \mathbf{D}$, with $P = P(L)$ and $Q = P(M)$, then

$$M^P \cong L^Q \cong D(P \times Q) \cong L \coprod M,$$

the coproduct of L and M in $\mathbf{D}$.

In 1968, E. T. Schmidt introduced the $M_3[D]$ *construction* (see [13] and the references in [21] for its significance). Although initially defined in universal algebraic terms, it turns out that $M_3[D]$ (where $D \in \mathbf{D}$ and M_3 is the five-element non-distributive modular lattice) is isomorphic to $M_3^{P(D)}$. Therefore $M[D]$ is defined in the literature to be $M^{P(D)}$ for any lattice M and any $D \in \mathbf{D}$. We have the

Proposition (E. T. Schmidt)**.** *Let $D \in \mathbf{D}$. Let L be the poset of all $(x, y, z) \in D^3$ such that $x \wedge y = y \wedge z = z \wedge x$. Then $L \cong M_3[D]$.*

As M_3 is a simple, complemented modular lattice with 3 atoms, this proposition led to Schmidt's posing the following

Problem (E. T. Schmidt [21], 1974)**.** *Is it possible to give a similar characterization for $M[D]$ if M is a finite simple complemented modular lattice?*

and the following

Conjecture (E. T. Schmidt [21], 1974)**.** *Let $D \in \mathbf{D}$. Let M be a finite simple complemented modular lattice with n atoms $p_1, \ldots, p_n$. Let L be the set of all n-tuples $(x_1, \ldots, x_n) \in D^n$ such that, if $p_j \leqslant p_k \vee p_l$ (p_j, p_k, p_l all distinct), then $x_j \wedge x_k = x_k \wedge x_l = x_l \wedge x_j$.*

Then $L \cong M[D]$.

We prove this conjecture below (Prop. 3.3).

In [19, §1], describing the $M_3[D]$ construction and Schmidt's proposition in the case where D is finite, Quackenbush wrote, "What is behind this curious duality in representing this modular lattice?"

An answer appeared in [8, Cor. 3.7]. (Cf. [14, §§2.3 and 3.2].) We summarize its conclusions below. In the theorem, we view our semilattices as commutative monoids with idempotent multiplication, so that we may consider the poset $\mathbf{Slat}(A, B)$ of semilattice homomorphisms preserving the identity element, even if A is a $\vee$-semilattice and B a $\wedge$-semilattice. (Note that the only part of the theorem we will actually use is the last; but the "duality" is most evident from the first, so we include it.)

Theorem (see [8])**.** *Let S be a $\vee$-semilattice with 0 and let $T \in \mathbf{D}$. View $(T, \wedge, 1_T)$ and $(T^\sigma, \cap, T)$ as $\wedge$-semilattices with 1. Then*

$$(S^\sigma)_\Sigma^{P(T)} \cong \mathbf{Slat}(S, T^\sigma)$$
$$(S^\sigma)_\Lambda^{P(T)} \cong \mathbf{Slat}(S, T)$$
$$(S^\sigma)^{P(T)} \cong \mathbf{Slat}^{\mathit{fin}}(S, T)$$

where $\mathbf{Slat}^{\mathit{fin}}(S, T)$ is the set of maps in $\mathbf{Slat}(S, T)$ with finite images.

(Ultimately this theorem is related to considering when the isomorphism

$$(L^P)^\sigma \cong (L^\sigma)^P.$$

holds. That is, when is the ideal poset of a power the power of an ideal poset. See [6], [9], [16].)

We will see that this theorem is intimately related to tensor products of semilattices: If S is a $\vee$-semilattice with 0, then

$$\mathbf{Slat}(S, \mathbf{2}) \cong (S^\sigma)^\partial,$$

the dual of the poset of ideals. Therefore, using the defining property of the tensor product $S \otimes T$ of $\vee$-semilattices with 0 (see the end of §2), we have

$$(S \otimes T)^{\sigma\partial} \cong \mathbf{Slat}(S \otimes T, \mathbf{2}) \cong \mathbf{Slat}\big(S, \mathbf{Slat}(T, \mathbf{2})\big),$$

or

$$\mathbf{Slat}(S, T^{\sigma\partial})^\partial \cong (S \otimes T)^\sigma,$$

If S and T are finite,

$$S \otimes T \cong \big(\mathbf{Slat}(S, T^\partial)\big)^\partial.$$

(See [1].)

One could ask what combinatorial properties are inherited by the lattice $S \otimes T$ when S and T are finite lattices. Two important candidate properties are *semimodularity* and *supersolvability* (see, for instance, [4], [22]). In 1985, Quackenbush made the following

Conjecture (Quackenbush [19], 1985)**.** *Let S be a finite modular lattice not having M_4 as a sublattice. Let T be a finite modular lattice. Then $S \otimes T$ is semimodular.*

We refute this conjecture below (Prop. 4.3).

The notion of *supersolvability* was introduced by R. P. Stanley in 1972.

Conjecture (Quackenbush [19], 1985)**.** *If B is a finite modular lattice, then $M_3 \otimes B$ is supersolvable if and only if B is distributive.*

We prove this conjecture below (Prop. 4.7).

2. Definitions and basic results

A general reference is [5]. Let P be a poset.

Let P^∂ denote the *dual* of P, where $x \leqslant y$ in P^∂ if and only if $x \geqslant y$ in P. Denote the least element of P by 0_P or 0, if it exists; denote the greatest element of P by 1_P or 1, if it exists. A poset with 0 and 1 is *bounded*. An element y *covers* an element x (denoted $x \lessdot y$) if $x < y$ and there is no $z \in P$ such that $x < z < y$; we say x is a *lower cover* of y and y is an *upper cover* of x. An element in a lattice L with a unique lower cover is *join-irreducible*; the set of all such is denoted $\mathcal{J}(L)$. If P has a 0, an *atom* is an upper cover of 0; if P has a 1, a *co-atom* is a lower cover of 1. If $x \leqslant y$, then $[x, y]$ denotes the *interval* $\{z \in P \mid x \leqslant z \leqslant y\}$. For $p \in P$, $\uparrow p = \{q \in P \mid p \leqslant q\}$. An *up-set* is a subset $U \subseteq P$ such that $\uparrow u \subseteq U$ for all $u \in U$.

A *chain* is a totally ordered poset (or subset of a poset). The *length* of a non-empty chain C is $|C| - 1$; the length of a non-empty poset is the largest length of a chain.

A non-trivial lattice is *simple* if the only homomorphic images are itself and $\mathbf{1}$. A lattice L is *distributive* if, for all $x, y, z \in L$,

$$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z).$$

A lattice L is *modular* if, for all $x, y, z \in L$ with $x \leqslant z$,

$$x \vee (y \wedge z) = (x \vee y) \wedge z.$$

A lattice is modular if and only if N_5 is not a sublattice; a lattice is distributive if and only if neither M_3 nor N_5 is a sublattice [5, 4.10]. The lattice consisting solely of 0, 1, and n atoms is denoted M_n.

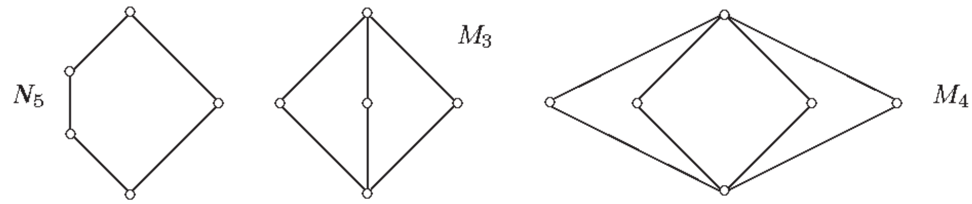

Figure 2.1. The lattices N_5, M_3, and M_4

A lattice L of finite length is *(upper) semimodular* if, for all $x, y \in L$, $x \wedge y \prec x$ implies $y \prec x \vee y$; dually we define *lower semimodularity*. A lattice of finite length is modular if and only if it is upper and lower semimodular (for instance [3, Chap. II, §8, Th. 16]). In a semimodular or lower semimodular lattice of finite length, all maximal chains have the same length (for instance [3, Chap. II, §8, Th. 14]).

A bounded lattice L is *complemented* if, for all $x \in L$, there exists $y \in L$ such that $x \wedge y = 0$ and $x \vee y = 1$. A modular lattice of finite length is complemented if and only if every join-irreducible is an atom (see [17, Lemma 4.83]).

The subspace lattice of the Fano plane is a finite complemented modular lattice of length 3 with 7 atoms (points) and 7 co-atoms (lines) with the property that every co-atom covers exactly 3 atoms and every atom is covered by exactly 3 co-atoms (see [17, §4.8]).

123 145 246 347 356 257 167

Figure 2.2. The lines of the Fano plane

A finite lattice is *supersolvable* if there is a maximal chain M such that, for any chain C, the sublattice generated by $C \cup M$ is distributive. All maximal chains have the same length in a supersolvable lattice, and every interval is supersolvable [22, Prop. 3.2(i)].

Let S, T, and U be $\vee$-semilattices with 0. Let $\mathbf{Slat}(S, T)$ denote the poset of all maps from S to T that preserve $\vee$ and 0. A map $f : S \times T \to U$ is a *bimorphism* if, for all $s \in S$ and $t \in T$, the maps $f(s, -) : T \to U$ and $f(-, t) : S \to U$ belong to $\mathbf{Slat}(T, U)$ and $\mathbf{Slat}(S, U)$ respectively. A *tensor product* of S and T, $S \otimes T$, is a $\vee$-semilattice with 0 together with a bimorphism $f : S \times T \to S \otimes T$ such that

(1) $f[S \times T]$ generates $S \otimes T$, and

(2) if $h : S \times T \to U$ is a bimorphism, then there exists $g \in \mathbf{Slat}(S \otimes T, U)$ such that $h = g \circ f$.

3. The $M[D]$ construction for a complemented modular lattice

In this section we prove the conjecture of E. T. Schmidt (Prop. 3.3; see §1). The basic idea of the proof is elementary, but for the sake of completeness we check all the details. (In response to a comment by a reader of a previous version of this manuscript, we note that resorting to the structure theorem for complemented modular lattices would not shorten the proof.)

Lemma 3.1. *Let S be a finite lattice and let T be an arbitrary lattice with 0, both viewed as $\vee$-semilattices with 0. Let $X := \{\big(f(j)\big)_{j\in\mathcal{J}(S)} \mid f \in \mathbf{Slat}(S,T)\}$ and*

$$Y := \left\{ (t_j)_{j\in\mathcal{J}(S)} \in T^{|\mathcal{J}(S)|} \,\middle|\, \text{for every } j \in \mathcal{J}(S) \text{ and every } K \subseteq \mathcal{J}(S)\backslash\{j\}, \; j \leqslant \bigvee K \text{ implies } t_j \leqslant \bigvee_{k\in K} t_k \right\}.$$

Then

(1) $X \cong \mathbf{Slat}(S,T)$,
(2) $X = Y$.

Proof. Clearly (1) holds and $X \subseteq Y$. Let $(t_j)_{j\in\mathcal{J}(S)} \in Y$. Define $f : S \to T$ by

$$f(a) = \bigvee_{\substack{j\in\mathcal{J}(S)\\ j\leqslant a}} t_j$$

for all $a \in S$. Suppose $a, b \in S$. Then

$$f(a \vee b) = f(a) \vee f(b) \vee \bigvee_{\substack{m\in\mathcal{J}(S)\\ m\leqslant a\vee b\\ m\nleqslant a,b}} t_m.$$

Let $N := \{n \in \mathcal{J}(S) \mid n \leqslant a \text{ or } n \leqslant b\}$. If $m \in \mathcal{J}(S)$ and $m \leqslant a \vee b$, but $m \notin N$, then $m \leqslant \bigvee N$, so $t_m \leqslant \bigvee_{n\in N} t_n = f(a) \vee f(b)$. Hence $f(a \vee b) = f(a) \vee f(b)$, and $f \in \mathbf{Slat}(S,T)$. Obviously, if $j, k \in \mathcal{J}(S)$ and $j < k$, then $t_j \leqslant t_k$; so $f(j) = t_j$ for all $j \in \mathcal{J}(S)$. $\Diamond$

Verbindungssatz [7, Cor. 4.8]**.** *Let M be a modular lattice of finite length. Let $p \in \mathcal{J}(M)$; let $a, b \in M$ be such that $p \leqslant a \vee b$ and $p \nleqslant a, b$. Then there exist $a', b' \in \mathcal{J}(M)$ such that $a' \leqslant a$; $b' \leqslant b$; and $p \leqslant a' \vee b'$.* $\Diamond$

(Note that in [7], the hypothesis $p \nleqslant a, b$ is omitted from the statement, without which the statement is trivially false.)

Lemma 3.2. *Let S, T, and Y be as in Lemma 3.1. Assume further that S is a complemented modular lattice. Let $(t_j)_{j\in\mathcal{J}(S)} \in T^{|\mathcal{J}(S)|}$. Then the following are equivalent:*

(1) $(t_j)_{j\in\mathcal{J}(S)} \in Y$,

(2) *for all distinct $j, k, l \in \mathcal{J}(S)$,*
$$j \leqslant k \vee l \text{ implies } t_j \vee t_k = t_k \vee t_l = t_l \vee t_j.$$

Proof. First we prove that (1) implies (2), recalling that every $j \in \mathcal{J}(S)$ is an atom (see §2). Let $j, k, l \in \mathcal{J}(S)$ be distinct and such that $j \leqslant k \vee l$. By semimodularity, $j \vee k = k \vee l = l \vee j$ and hence, by the definition, $t_j \leqslant t_k \vee t_l$, $t_k \leqslant t_j \vee t_l$, and $t_l \leqslant t_j \vee t_k$. Hence $t_j \vee t_k = t_k \vee t_l = t_l \vee t_j$.

Now we show that (2) implies (1). We prove by induction on n that, for all $j \in \mathcal{J}(S)$ and all $K \subseteq \mathcal{J}(S) \setminus \{j\}$ such that $|K| = n$, $j \leqslant \bigvee K$ implies $t_j \leqslant \bigvee_{k\in K} t_k$.

This is vacuously true if $n = 0$ or 1. (For $n = 1$, recall that the join-irreducibles form an antichain.) If $n = 2$ – say $K = \{k, l\}$ – then $j \leqslant k \vee l$ implies $t_j \leqslant t_k \vee t_l$ by (2).

Now assume $n \geqslant 3$. Let $j \in \mathcal{J}(S)$. Let $K \subseteq \mathcal{J}(S) \setminus \{j\}$ be such that $j \leqslant \bigvee K$ where $|K| = n$. We may assume that $j \nleqslant \bigvee L$ for every $L \subsetneq K$.

Take any $a \in K$ and let $b = \bigvee (K \setminus \{a\})$. By the Verbindungssatz, there exist $k, l \in \mathcal{J}(S)$ such that $k \leqslant a$; $l \leqslant b$; and $j \leqslant k \vee l$. Clearly $k = a$. Also, j, k, and l are distinct; so by (2), $t_j \vee t_k = t_k \vee t_l = t_l \vee t_j$. By the induction hypothesis, $t_l \leqslant \bigvee_{i\in K\setminus\{k\}} t_i$ so
$$t_j \leqslant \bigvee_{i\in K} t_i. \quad \Diamond$$

Proposition 3.3. *Let S be a finite complemented modular lattice with exactly n distinct atoms $p_1, \ldots, p_n$. Let T be a bounded distributive lattice. Then $S[T]$ is isomorphic to the set of n-tuples $(t_1, \ldots, t_n) \in T^n$ such that, for all distinct $j, k, l \in \{1, \ldots, n\}$, $p_j \leqslant p_k \vee p_l$ implies $t_j \wedge t_k = t_k \wedge t_l = = t_l \wedge t_j$.*

Note. We do not need to insist, as Schmidt does, that S be simple. Also, in E. T. Schmidt's original conjecture [21], he does not explicitly state that p_j, p_k, p_l must be *distinct*, although this clearly must be specified.

Proof. By a corollary of the theorem from [8] cited in §1,
$$S[T] = S^{P(T)} \cong \mathbf{Slat}(S, T^\partial)^\partial.$$

Remember that here we are viewing **Slat** as picking out the $\vee, 0$-semilattice maps, whereas in the aforementioned corollary **Slat** picks out maps from a $\vee, 0$-semilattice to a $\wedge, 1$-semilattice. The result follows from Lemmas 3.1 and 3.2 by considering the (order) duals of the relevant lattices. $\Diamond$

This solves the problem and proves the conjecture of E. T. Schmidt.

4. The semimodularity and supersolvability of some semilattice tensor products

In this section we refute a conjecture and prove another conjecture made by Quackenbush (Props. 4.3 and 4.7; see §1).

In Lemmas 4.1, 4.2, and Prop. 4.3, let S be the subspace lattice of the Fano plane, with atoms (points) $a_1, a_2, a_3, a_4, a_5, a_6, a_7$. Let the atoms of M_7 be labelled A, B, C, D, E, F, G. Let $N \cong \mathbf{Slat}(S, M_7)$.

Lemma 4.1. *The lattice N is neither semimodular nor lower semimodular.*

Proof. We show that N has a maximal chain of length 6 and a maximal chain of length at least 8, thus violating semimodularity. Let us assume that $\{a_1, a_2, a_3\}$ is a line in the Fano plane. By Lemma 3.2, we may assume that

$$N = \big\{(t_1, \ldots, t_7) \in (M_7)^7 \mid \text{if } \{a_j, a_k, a_l\} \text{ is a line, then } t_j \vee t_k = t_k \vee t_l = t_l \vee t_j\big\}.$$

The following is a maximal chain in N: $n_0 = 0000000$, $n_1 = 000AAAA$, $n_2 = 0AAAAAA$, $n_3 = AAAAAAA$, $n_4 = AAA1111$, $n_5 = A111111$, $n_6 = 1111111$. We have $n_0 \lessdot n_1$ since, if any two points go to 0, the whole line must go to 0; $n_1 \lessdot n_2$ for the same reason; $n_2 \lessdot n_3$ is obvious; $n_3 \lessdot n_4$ since if any two points go to A and the third point on the line goes to A or 1, the whole line must go to A; $n_4 \lessdot n_5$ for the same reason; and $n_5 \lessdot n_6$ is obvious.

The following is also a chain in N:

$$0000000, ABCDEFG, ABCDEF1, ABCDE11,$$
$$ABCD111, ABC1111, AB11111, A111111, 1111111. \quad \Diamond$$

Lemma 4.2. *The lattice M_4 is not a sublattice of S.*

Proof. If it were, then (possibly considering the dual of S) at least two of the atoms of M_4 would be atoms of S, in which case, all four atoms would be atoms of S. But no line contains four points. $\Diamond$

Proposition 4.3. *There exist a finite modular lattice S not having M_4 as a sublattice and a finite modular lattice T such that $S \otimes T$ is not semimodular.*

Proof. Let $T = M_7$ and use the fact that $S \otimes T \cong \mathbf{Slat}(S, T^\partial)^\partial$ (see §1) as well as Lemmas 4.1 and 4.2. $\Diamond$

Lemma 4.4. *Let S and T be finite lattices. Let $x, y \in T$ be such that $x \leqslant y$. Then*

$$\bigl\{f \in \mathbf{Slat}(S,T) \bigm| f(a) \in [x,y] \text{ for all } a \in S \setminus \{0\}\bigr\}$$

is an interval in the lattice $\mathbf{Slat}(S,T)$ isomorphic to $\mathbf{Slat}(S,[x,y])$.

Proof. For $z \in T$, let $g_z : S \to T$ be defined for all $s \in S$ by

$$g_z(s) = \begin{cases} z & \text{if } s > 0_S, \\ 0_T & \text{if } s = 0_S. \end{cases}$$

Then $g_z \in \mathbf{Slat}(S,T)$. The interval in question is $[g_x, g_y]$. $\Diamond$

Lemma 4.5. *Let $k \geqslant 3$. Then $\mathbf{Slat}(M_3, M_k)$ is not supersolvable.*

Proof. (Cf. [19, Prop. 19].) We use Lemma 3.2 and show that every atom of Y has two comparable complements in some interval. The atoms of Y are of two types:

$$AA0$$
$$BCA$$

where A, B, and C are distinct atoms of M_k. But $\{000, AA0, BCA, 1CA, 11A\}$ is a sublattice isomorphic to N_5, as is $\{000, BCA, AA0, AAA, 11A\}$ (Fig. 4.1). $\Diamond$

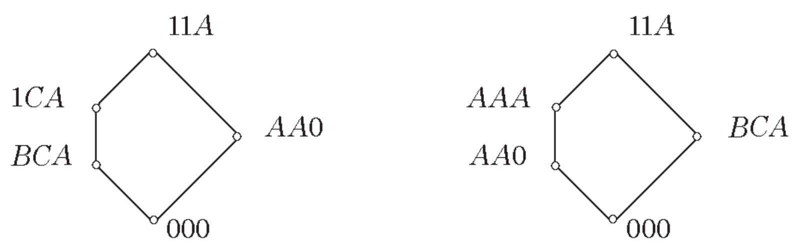

Figure 4.1. Sublattices of $Y \cong \mathbf{Slat}(M_3, M_k)$ isomorphic to N_5

Lemma 4.6. *Let T be a finite modular lattice that is not distributive. Then there exist $x, y \in T$ such that $x \leqslant y$ and $[x,y] \cong M_k$ for some $k \geqslant 3$.*

Proof. See [2, Chap. IX, §1, Cor. 2] (note the typographical mistake in the text); cf. [12, §2]. $\Diamond$

Proposition 4.7. *Let B be a finite modular lattice. Then the following are equivalent:*

(1) *B is distributive,*
(2) *$M_3 \otimes B$ is modular,*
(3) *$M_3 \otimes B$ is supersolvable.*

Note. The equivalence of (1) and (2) is [19, Lemma 8] (for B a general bounded modular lattice). Also, supersolvability in the literature is defined for *finite* lattices (or, at least, lattices of finite length). In the conjecture of [19], this hypothesis was not explicitly stated, an oversight confirmed by Quackenbush [20].

Proof. If B is distributive, then $M_3 \otimes B \cong M_3^P$ for some finite poset P (see §1), so it is modular. If $M_3 \otimes B$ is modular, then it is supersolvable. If $M_3 \otimes B$ is supersolvable, but B is not distributive, then there exist $x, y \in B$ such that $x \leqslant y$ and $[x, y] \cong M_k$ for some $k \geqslant 3$ (Lemma 4.6). By Lemma 4.4, $\mathbf{Slat}(M_3, M_k)$ is isomorphic to an interval in $\mathbf{Slat}(M_3, B^\partial)$, so it is supersolvable. This contradicts Lemma 4.5. $\Diamond$

(A reader of a previous version of this manuscript has pointed out that M_3 can be replaced by any finite modular, non-distributive lattice. This reader also made a conjecture which we have not considered, namely, that if every simple complemented interval of M is isomorphic to either M_3 or $\mathbf{2}$, then $M \otimes B$ is semimodular for every finite modular lattice B.)

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