Journal of Combinatorial Theory, Series A **90**, 123–147 (2000)
doi:10.1006/jcta.1999.3021, available online at http://www.idealibrary.com on IDEAL®

Quasi-Differential Posets and Cover Functions of Distributive Lattices

I. A Conjecture of Stanley

Jonathan David Farley

Department of Mathematics, Vanderbilt University, Nashville, Tennessee 37240
E-mail: farley@math.vanderbilt.edu

Communicated by Gian-Carlo Rota

Received January 6, 1999

A distributive lattice L with 0 is *finitary* if every interval is finite. A function $f: \mathbb{N}_0 \to \mathbb{N}_0$ is a *cover function* for L if every element with n lower covers has $f(n)$ upper covers. In this paper, all finitary distributive lattices with non-decreasing cover functions are characterized. A 1975 conjecture of Richard P. Stanley is thereby settled.

Key Words: differential poset; Fibonacci lattice; distributive lattice; (partially) ordered set; cover function.

A. PRELIMINARIES

1. *Stanley's Conjecture*

Fibonacci numbers, aptly enough, are a recurring phenomenon in mathematics; they even appear in lattice theory. Stanley has investigated certain distributive lattices related to the Fibonacci numbers in [2].

He notes that many of these lattices have the following property: if two elements have the same number (n) of immediate predecessors, then they have the same number ($f(n)$) of immediate successors. Hence one may define a *cover function* $f: \mathbb{N}_0 \to \mathbb{N}_0$, where $\mathbb{N}_0 = \{0, 1, 2, ...\}$.

In his 1975 paper, Stanley conjectures that the only *non-decreasing* cover functions are the constant functions and functions of the form $f(n) = n + k$ for some constant k. We settle this conjecture by characterizing all non-decreasing cover functions and the corresponding lattices (Theorem 11.1).

In the rest of Part A we shall define our terms and state the conjecture precisely (Section 3). Then we shall present background material more directly related to the conjecture and give some basic examples.

In Part B we shall settle the conjecture by doing a case-by-case analysis of all the possible non-decreasing cover functions.

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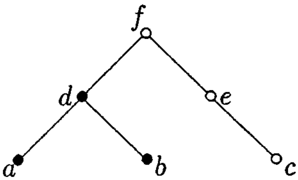

$Q = \{a, b, d\} = \downarrow d$

$\mathrm{Irr}(P) = \{e\}$

$\mathrm{LC}(f) = \{d, e\}$

FIG. 2.1. A down-set Q of P.

2. *General Definitions, Notation, and Basic Theory*

For basic facts and notation, see [1, 3].

Let P be a poset. We denote the least element by 0_P or 0 if it exists.

Let p, $q \in P$. We say p is a *lower cover* of q and q is an *upper cover* of p (denoted $p \lessdot q$) if $p < q$ and there is no $r \in P$ such that $p < r < q$. We denote the set of lower covers of p by $\mathrm{LC}(p)$. An element is (*join*-) *irreducible* if it has a unique lower cover. Let $\mathrm{Irr}(P)$ denote the poset of irreducibles of P.

A subset $Q \subseteq P$ is a *down-set* (or *order ideal*) if $p \in P$, $q \in Q$, and $p \leqslant q$ imply $p \in Q$ (Fig. 2.1).

The family of *finite* down-sets of P is denoted $\mathcal{O}_{\mathrm{f}}(P)$. For $R \subseteq P$,

$$\downarrow R = \{p \in P \mid p \leqslant r \text{ for some } r \in R\};$$

if R is a singleton $\{r\}$, we simply write $\downarrow r$, and $\mathring{\downarrow} r$ denotes $(\downarrow r) \backslash \{r\}$. (Note that $\downarrow R$ is a down-set.)

Let P and Q be posets. The *disjoint sum* of P and Q, $P + Q$, is the poset with underlying set $P \cup Q$ such that p and q are incomparable for all $p \in P$ and $q \in Q$ (Fig. 2.2). The *ordinal sum* of P and Q, $P \oplus Q$, is the poset on $P \cup Q$ such that $p < q$ for all $p \in P$ and $q \in Q$ (Fig. 2.3).

If P has a greatest element and Q a least element, the *coalesced ordinal sum*, $P \boxplus Q$, is the poset obtained by identifying these two elements (Fig. 2.4).

The *direct product* $P \times Q$ is the set of pairs (p, q) ordered coordinatewise: $(p, q) \leqslant (p', q')$ if $p \leqslant p'$ and $q \leqslant q'$ $(p, p' \in P,\ q, q' \in Q)$—see Figs. 2.5a and 2.5b.

An *antichain* is a poset in which distinct elements are incomparable; a *chain* is a totally ordered set. For $n \in \mathbb{N}_0$, the n-element chain is denoted $\mathbf{n}$ (Fig. 2.6).

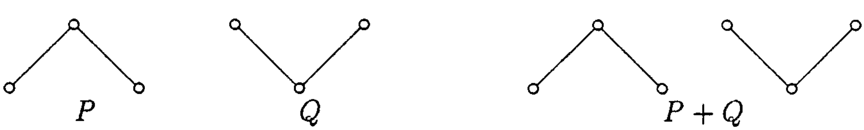

FIG. 2.2. The disjoint sum.

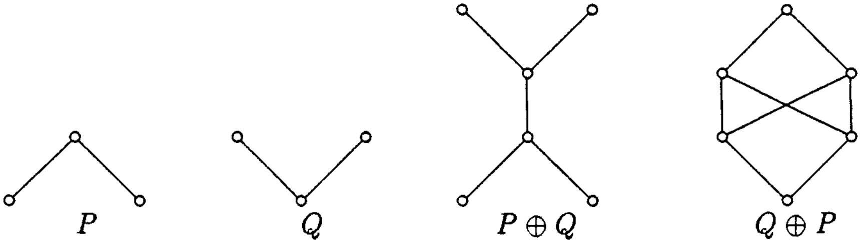

FIG. 2.3. The ordinal sum.

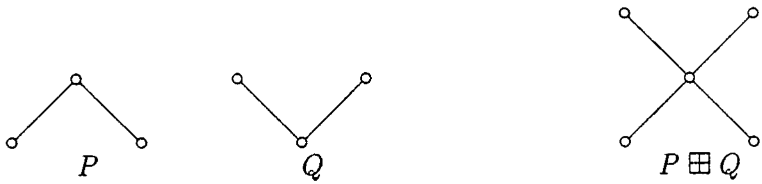

FIG. 2.4. The coalesced ordinal sum.

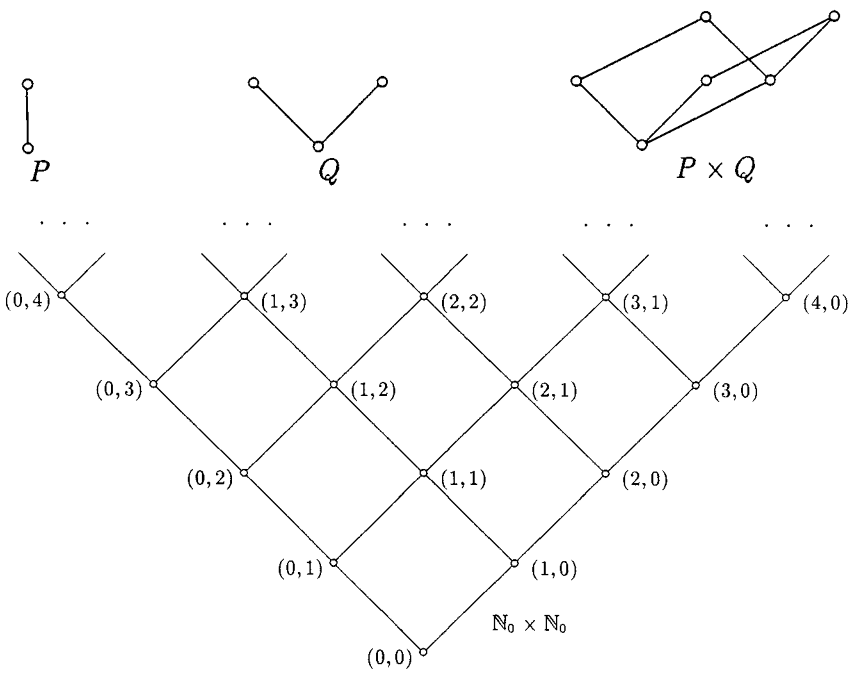

FIG. 2.5. (a) Direct products. (b) Direct products.

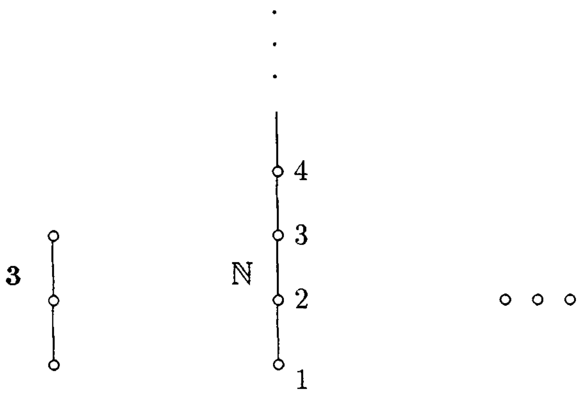

FIG. 2.6. Chains and an antichain.

A lattice L is *finitary* if it has a 0 and $\downarrow a$ is finite for all $a \in L$. It is well known that a finitary distributive lattice may be identified with $\mathcal{O}_{\mathrm{f}}(P)$ where $P = \operatorname{Irr} L$ ([3, 3.4.3]).

If we do simply say that $L = \mathcal{O}_{\mathrm{f}}(P)$, then $I \lessdot J$ in L if and only if $I = J \backslash \{j\}$ for a maximal element $j \in J$ (now viewed as a subposet of P).

For posets P and Q, $\mathcal{O}_{\mathrm{f}}(P + Q) \cong \mathcal{O}_{\mathrm{f}}(P) \times \mathcal{O}_{\mathrm{f}}(Q)$, and, if P is finite, $\mathcal{O}_{\mathrm{f}}(P \oplus Q) \cong \mathcal{O}_{\mathrm{f}}(P) \boxplus \mathcal{O}_{\mathrm{f}}(Q)$. In particular, $\mathcal{O}_{\mathrm{f}}(\mathbf{1} \oplus Q) \cong \mathbf{1} \oplus \mathcal{O}_{\mathrm{f}}(Q)$. (See Figs. 2.7 and 2.8 and [1, Chap. 8].)

Let Y denote *Young's lattice* (a lattice of great interest to combinatorialists). It is the poset of sequences $(a_1, a_2, \ldots) \in \mathbb{N}_0^{\omega}$ with finitely many non-zero coordinates such that $a_1 \geqslant a_2 \geqslant \cdots$. We will identify Young's lattice with $\mathcal{O}_{\mathrm{f}}(\mathbb{N}_0 \times \mathbb{N}_0)$ (Fig. 2.9).

3. *Definition of Cover Functions and Known Results*

Let L be a finitary distributive lattice. A function $f\colon \mathbb{N}_0 \to \mathbb{N}_0$ is a *cover function* for L if every element with (exactly) n lower covers has (exactly) $f(n)$ upper covers. (The definition comes from [2, Sect. 3; 3, p. 157]; cf. the definition of *differential posets* in [4].)

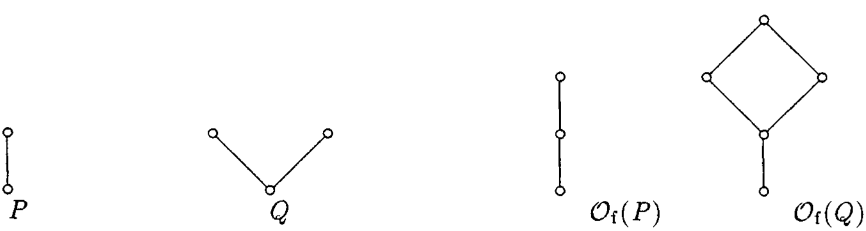

FIG. 2.7. The lattice of down-sets.

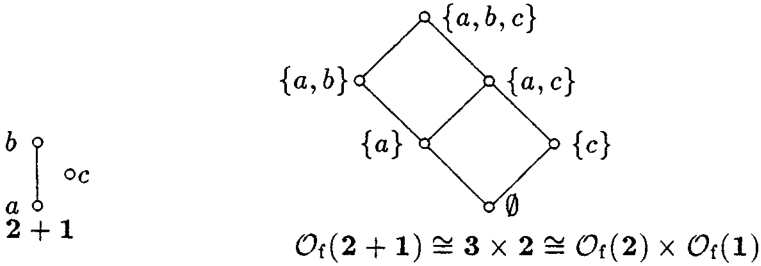

FIG. 2.8. The lattice of down-sets.

The first three examples come from [2].

EXAMPLE 3.1. For $k \in \mathbb{N}$, the constant function $f(n) = k$ $(n \in \mathbb{N}_0)$ is a cover function for $\mathbb{N}_0^k$ (Figs. 3.1a and 3.1b).

[We note that $f(n)$ could take any value for $n > k$.]

EXAMPLE 3.2. For $k \in \mathbb{N}$, the function $f(n) = k + n$ $(n \in \mathbb{N}_0)$ is a cover function for Y^k. (See Lemma 4.8.)

EXAMPLE 3.3. For $k \in \mathbb{N}_0$, any function $f\colon \mathbb{N}_0 \to \mathbb{N}_0$ with $f(n) = k - n$ $(0 \leqslant n \leqslant k)$ is a cover function for $\mathbf{2}^k$ (Fig. 3.2).

In fact, we have:

PROPOSITION 3.4 [2, Sect. 3, Proposition 2]. *If L is a finite distributive latttice with a cover function, then $L \cong \mathbf{2}^r$ for some $r \in \mathbb{N}_0$.*

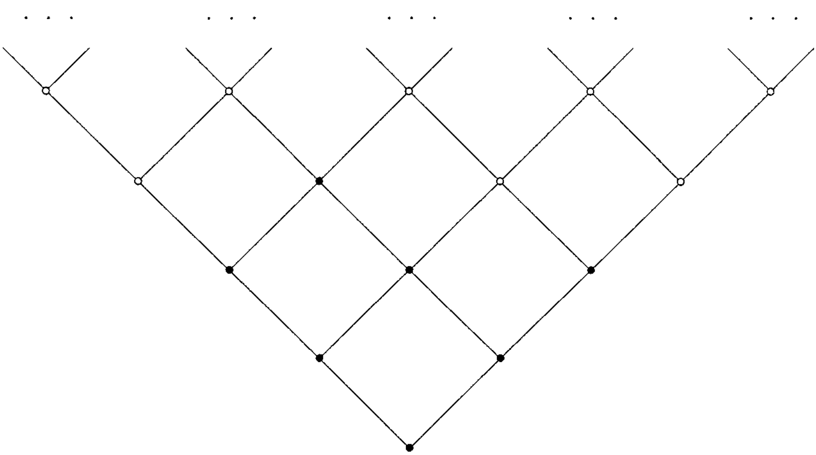

FIG. 2.9. An *element* of Young's lattice.

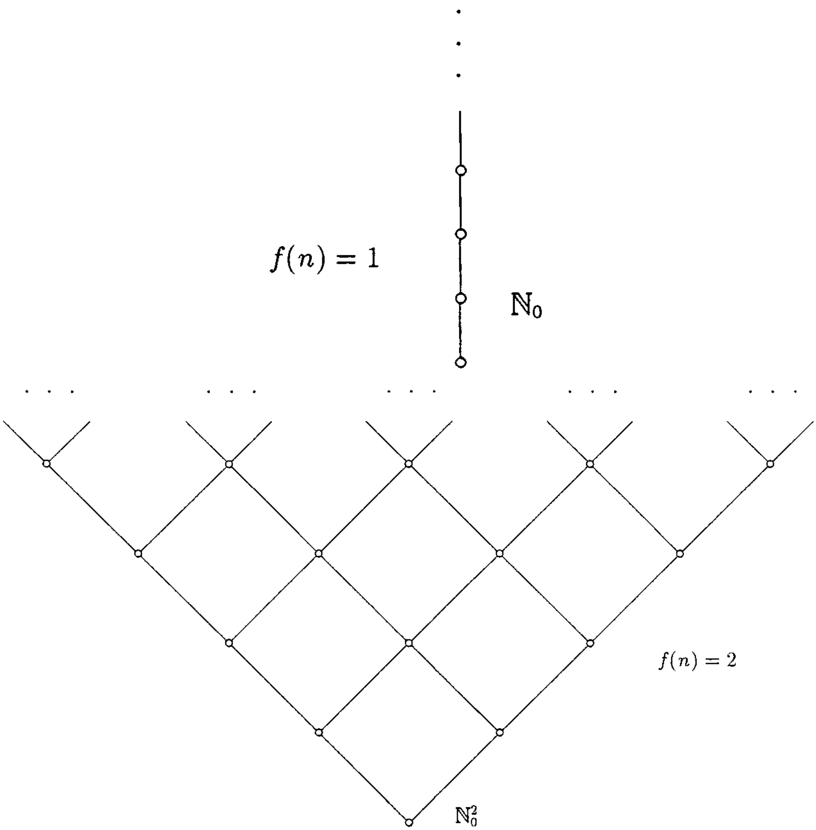

FIG. 3.1. (a) Cover functions. (b) Cover functions.

Aside. It appears to us that there needs to be an additional step in the proof of the statement that appears in [2]. For it proceeds by assuming that $P = \mathrm{Irr}(L)$ has r maximal elements $x_1, ..., x_r$, and that the down-set $I \backslash \{x_1, ..., x_r\}$ has s maximal elements. Then down-sets $I_k = I \cup \{x_1, ..., x_k\}$ are constructed for $1 \leqslant k \leqslant r$. "Then each I_k is an order ideal of P, and the

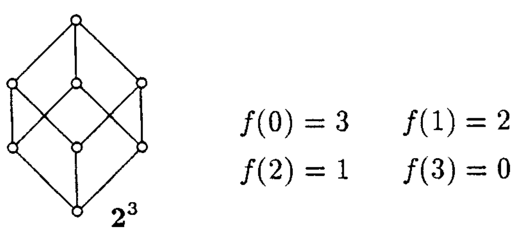

FIG. 3.2. A cover function.

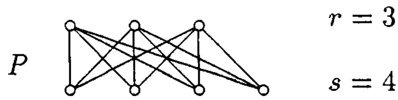

FIG. 3.3. An illustration of the argument.

number of maximal elements of I_k is at most one more than the number of maximal elements of I_{k-1}. Since I_1 has $\leqslant s$ maximal elements and I_r has r maximal elements, some I_k has s maximal elements."

The assumption seems to be that $s \leqslant r$. Figure 3.3, however, illustrates the above set-up (*sans* the existence of a cover function) in which the conclusion of the quoted statement does not hold.

We have constructed the following examples:

EXAMPLE 3.5. For $k \geqslant 2$, the function

$$f(n) = \begin{cases} k-n & \text{if } \; 0 \leqslant n < k, \\ k & \text{if } \; n = k, \\ * & \text{otherwise,} \end{cases}$$

where $n \in \mathbb{N}_0$, is a cover function for $\boxplus_{i=1}^{\infty} \mathbf{2}^k$ (Fig. 3.4).

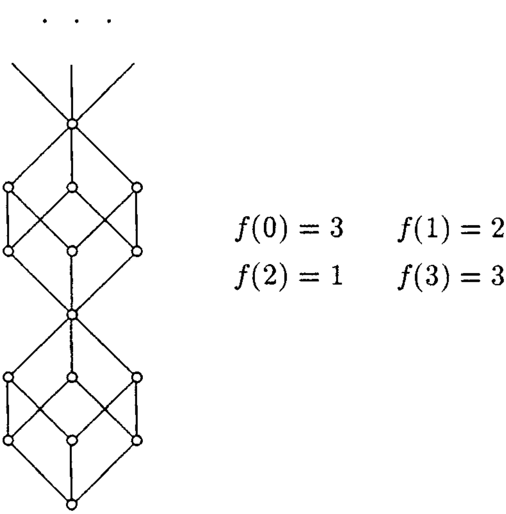

FIG. 3.4. A cover function.

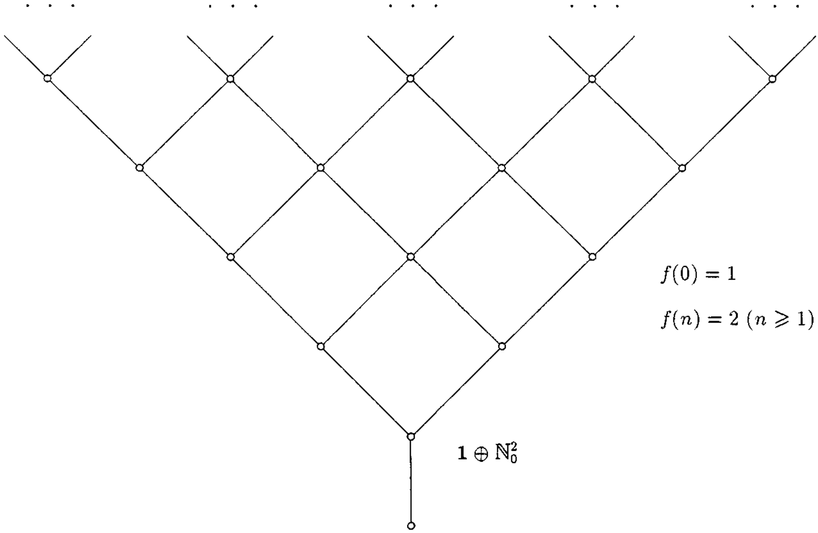

FIG. 3.5. Cover functions.

EXAMPLE 3.6. For $k \geqslant 2$, the function

$$f(n) = \begin{cases} 1 & \text{if} \quad n = 0, \\ k & \text{if} \quad 1 \leqslant n \leqslant k, \\ * & \text{otherwise,} \end{cases}$$

where $n \in \mathbb{N}_0$, is a cover function for $\mathbf{1} \oplus \mathbb{N}_0^k$ (Fig. 3.5).

EXAMPLE 3.7. The poset $L = Y \backslash \{0_Y\}$ is still a finitary distributive lattice [with $\mathrm{Irr}(L) \cong (\mathbb{N}_0 \times \mathbb{N}_0) \backslash \{(0, 0)\}$], and it has cover function

$$f(n) = \begin{cases} 2 & \text{if} \quad n = 0, \\ n+1 & \text{if} \quad n \geqslant 1, \end{cases}$$

where $n \in \mathbb{N}_0$.

EXAMPLE 3.8. Another "sporadic" example is the lattice $L = \mathbf{2}^2 \oplus \mathbb{N}$, which has cover function

$$f(n) = \begin{cases} 2 & \text{if} \quad n = 0, \\ 1 & \text{if} \quad 1 \leqslant n \leqslant 2, \\ * & \text{otherwise,} \end{cases}$$

where $n \in \mathbb{N}_0$ (Fig. 3.6).

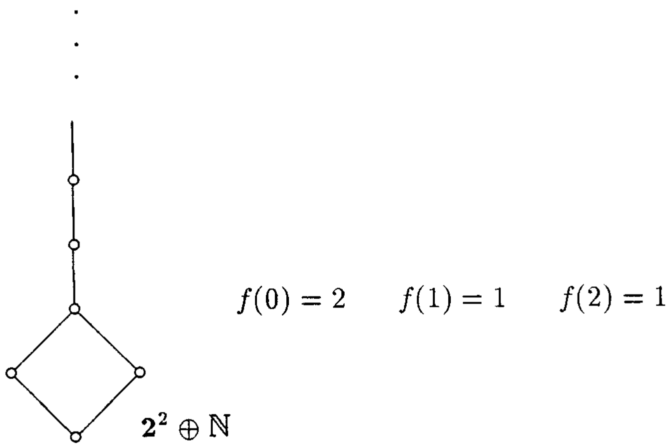

FIG. 3.6. A cover function.

Even though we have seen that a given lattice L may have more than one cover function, any two lattices with the same cover function must be isomorphic:

PROPOSITION 3.9 [2, Sect. 3; 3, pp. 157, 180]. *There is at most one finitary distributive lattice with a given cover function* (*up to isomorphism*).

In [2], Stanley states the following:

Conjecture (Stanley, [2]). "We in fact conjecture that if L [a finitary distributive lattice] has a ... *non-decreasing* cover function $f(n)$ (i.e., $f(i+1) \geqslant f(i)$), with $f(0) > 0$, then $f(n) = a$ or $f(n) = n + a$."

In [2, Sect. 3], Stanley proves that no function of the form $f(n) = an + b$ is a cover function if $|a| \geqslant 2$. He uses an interesting result.

PROPOSITION 3.10 [2, Sect. 3; 3, pp. 157, 179–180]. *Let L be a finitary distributive lattice with finitely many elements of each rank. Let $u(i, j)$ ($v(i, j)$) be the number of elements of rank i with exactly j lower* (*upper*) *covers.*

Then for $i \geqslant j \geqslant 0$,

$$\sum_{k=0}^{\infty} u(i, k) \binom{k}{j} = \sum_{k=0}^{\infty} v(i-j, k) \binom{k}{j}.$$

Strictly speaking, because of the freedom we have in choosing some cover functions (see Examples 3.1, 3.3, 3.5, 3.6, and 3.8), this conjecture is false. In light of Proposition 3.9 (and Examples 3.1–3.3), however, it is clear that Stanley means the following:

Conjecture′. If L is a finitary distributive lattice with a non-decreasing cover function, then $L \cong \mathbb{N}_0^k$ or $L \cong Y^k$ for some $k \in \mathbb{N}_0$.

Examples 3.6 and 3.7 show that this conjecture, too, is false; yet it is *morally* true. In Part B we prove the following (Theorem 11.1):

THEOREM. *If L is a finitary distributive lattice with a non-decreasing cover function, then one of the following holds*:

(1) $L \cong \mathbb{N}_0^k$ $(k \geqslant 1)$;

(2) $L \cong Y^k$ $(k \geqslant 1)$;

(3) $L \cong \mathbf{1} \oplus \mathbb{N}_0^k$ $(k \geqslant 2)$;

(4) $L \cong Y \backslash \{0_Y\}$;

(5) $L \cong \mathbf{1}$.

B. RESOLUTION OF STANLEY'S CONJECTURE

In Part B, L will denote a finitary distributive lattice with cover function $f\colon \mathbb{N}_0 \to \mathbb{N}_0$. Let $P = \mathrm{Irr}(L)$ and let $x_1, ..., x_m$ be its set of minimal elements. (It is clear that $m = f(0)$.)

We identify L with $\mathcal{O}_f(P)$.

4. *Useful Lemmas*

We will use the following key lemmas repeatedly; they mostly follow from the characterization of the cover relation in $\mathcal{O}_f(P)$ given in Section 2.

LEMMA 4.1. *Let M be a finitary distributive lattice and let N be the finitary distributive lattice $\mathbf{1} \oplus M$.*

(1) *If M has cover function $g\colon \mathbb{N}_0 \to \mathbb{N}_0$ such that $g(0) = g(1)$, define a cover function $h\colon \mathbb{N}_0 \to \mathbb{N}_0$ for N by*

$$h(n) = \begin{cases} 1 & \text{if} \quad n = 0, \\ g(n) & \text{if} \quad n \geqslant 1 \end{cases}$$

$(n \in \mathbb{N}_0)$.

(2) *If N has a cover function $h\colon \mathbb{N}_0 \to \mathbb{N}_0$, define for M a cover function $g\colon \mathbb{N}_0 \to \mathbb{N}_0$ such that $g(0)=g(1)$ by*

$$g(n)=\begin{cases} h(1) & \text{if } \ n=0,\\ h(n) & \text{if } \ n\geqslant 1\end{cases}$$

$(n \in \mathbb{N}_0)$.

LEMMA 4.2. *Let $n \in \mathbb{N}_0$. An element $I \in \mathcal{O}_f(P)$ has (exactly) n lower covers in L if and only if I has n maximal elements in P.*

LEMMA 4.3. *If $I = \downarrow A$ where $A \subseteq P$ is an n-element antichain, then there are (exactly) $f(n)$ elements $p \in P\backslash I$ such that $\mathrm{LC}(p) \subseteq I$.*

LEMMA 4.4. *Let $q \in P$; let $A = \{p \in P\backslash\downarrow q \mid \mathrm{LC}(p) \subseteq \mathring{\downarrow} q\}$, and let $B = \{p \in P\backslash \downarrow q \mid \mathrm{LC}(p) \subseteq \downarrow q\}$.*
Then:

(1) $A \subseteq B$;

(2) *for all $r \in P$, $r \in B\backslash A$ if and only if $r \in \mathrm{Irr}(P)$ and $q \prec r$.*

COROLLARY 4.5. *Assume that $f(0)=1$. Then P has a down-set Q isomorphic to $\{0_P\} \oplus \sum_{i=1}^{f(1)} \mathbb{N}$ such that $Q\backslash\{0_P\} \subseteq \mathrm{Irr}(P)$ and every element in $Q\backslash\{0_P\}$ has a unique irreducible upper cover in P (namely, its upper cover in Q).*

Proof. Let $y_1, ..., y_{f(1)}$ be the upper covers of 0_P in P. (Without loss of generality, $f(1) \geqslant 1$.) Let $q = y_1$ in Lemma 4.4: Then A has $f(1)-1$ elements, namely, $y_2, ..., y_{f(1)}$, and B has $f(1)$ elements. Hence y_1 has a unique irreducible upper cover in P, y_1'.

Similarly, if $n \geqslant 1$ and $y_1 \prec y_1' \prec \cdots \prec y_1^{(n)}$ where $y_1^{(i+1)}$ is the unique irreducible upper cover of $y_1^{(i)}$ in P $(0 \leqslant i < n)$, let $r = y_1^{(n)}$ in Lemma 4.4. Since $y_1^{(i)}$ is irreducible in P, $\{y_1^{(i)}, y_2, ..., y_{f(1)}\}$ is an antichain $(1 \leqslant i \leqslant n)$. Thus $A = \{y_2, ..., y_{f(1)}\}$ has $f(1)-1$ elements and B has $f(1)$ elements, so $y_1^{(n)}$ has a unique irreducible upper cover, $y_1^{(n+1)}$.

By induction, we construct a subposet with the desired properties. ∎

COROLLARY 4.6. *If $f(0)=f(1)$, then P has a down-set Q isomorphic to $\sum_{i=1}^{f(1)} \mathbb{N}$ such that every element of Q has a unique irreducible upper cover in P (namely, its upper cover in Q).*

Proof. This corollary follows from Lemma 4.1 and Corollary 4.5. ∎

LEMMA 4.7. *If $f(0) \geqslant 2$, then $f(0)-1 \leqslant f(1) \leqslant f(0)+1$.*

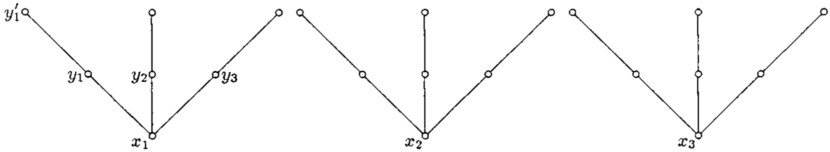

FIG. 4.1. An impossible scenario ($f(0)=3$, $f(1)=5$).

Proof. The first inequality holds since $\{x_1\}$ has, in L, at least the $f(0)-1$ upper covers $\{x_1, x_i\}$ $(2 \leqslant i \leqslant f(0))$.

Suppose for a contradiction that $f(1) \geqslant f(0)+2$. Let $q=x_1$ in Lemma 4.4. Then $A=\{x_2, ..., x_{f(0)}\}$ and x_1 has exactly $f(1)-(f(0)-1)=f(1)-f(0)+1$ irreducible upper covers in P, $y_1, y_2, ..., y_{f(1)-f(0)+1}$. (Note that $f(1)-f(0)+1 \geqslant 3$.) Hence $\{x_1, x_2\}$ has at least $2[f(1)-f(0)+1]+(f(0)-2)=2f(1)-f(0)$ upper covers in L, the first batch obtained from the irreducible upper covers of x_1 and x_2, the second batch being $\{x_1, x_2, x_i\}$ $(3 \leqslant i \leqslant f(0))$. (So $f(2) \geqslant 2f(1)-f(0)$.)

Now $\downarrow y_1$ has at least $(f(0)-1)+(f(1)-f(0))=f(1)-1$ upper covers in L, namely, $\downarrow y_1 \cup \{x_i\}$ $(2 \leqslant i \leqslant f(0))$ and $\downarrow y_1 \cup \downarrow y_j$ $(2 \leqslant j \leqslant f(1)-f(0)+1)$. Let $q=y_1$ in Lemma 4.4; then y_1 has exactly 1 irreducible upper cover in P, y_1' (Fig. 4.1).

Consider $\downarrow y_1 \cup \downarrow y_2$. Some of its upper covers in L are $\downarrow y_1 \cup \downarrow y_2 \cup \{x_i\}$ $(2 \leqslant i \leqslant f(0))$, $\downarrow y_1 \cup \downarrow y_2 \cup \downarrow y_j$ $(3 \leqslant j \leqslant f(1)-f(0)+1)$, $\downarrow y_1' \cup \downarrow y_2$ and $\downarrow y_1 \cup \downarrow y_2'$; these number $(f(0)-1)+[(f(1)-f(0)+1)-2]+2=f(1)$. Hence there are exactly $f(2)-f(1) \geqslant f(1)-f(0) \geqslant 2$ elements $z \in P$ such that $\mathrm{LC}(z)=\{y_1, y_2\}$—let z_1, z_2 be two such (Fig. 4.2).

Then $\downarrow z_1$ has *more* than $f(1)$ upper covers in L, namely, $\downarrow z_1 \cup \{x_i\}$ $(2 \leqslant i \leqslant f(0))$, $\downarrow y_j \cup \downarrow z_1$ $(3 \leqslant j \leqslant f(1)-f(0)+1)$, $\downarrow y_1' \cup \downarrow z_1$, $\downarrow y_2' \cup \downarrow z_1$, and $\downarrow z_1 \cup \downarrow z_2$, a contradiction.

Hence $f(1) \leqslant f(0)+1$. ∎

LEMMA 4.8. *Let $g\colon \mathbb{N}_0 \to \mathbb{N}_0$ be the function $g(n)=n+1$ $(n \in \mathbb{N}_0)$. Then g is a cover function for Y.*

Proof. See Lemma 5.3. ∎

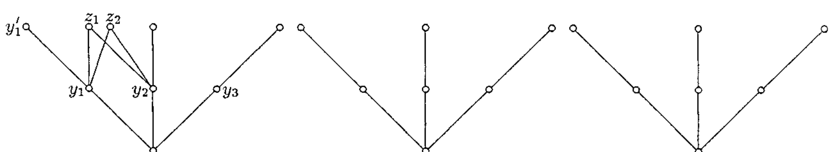

FIG. 4.2. An impossible scenario ($f(0)=3$, $f(1)=5$).

5. $f(0)=1$; $f(1)=2$; $f(2)\neq 2$

LEMMA 5.1. *The poset P has a down-set Q isomorphic to the subposet*

$$\{(m,n)\in\mathbb{N}_0\times\mathbb{N}_0 \mid m=0 \text{ or } n=0\}$$

and every element of $Q\backslash\{0_P\}$ has a unique irreducible upper cover in P.

Proof. The result follows from Corollary 4.5 and the fact that $f(1)=2$. ∎

LEMMA 5.2. *Let Q be the set of Lemma 5.1. Then*

(1) *There is a unique $y\in P$ such that* $\mathrm{LC}(y)=\{(0,1),(1,0)\}$;

(2) $f(2)=3$.

Proof. Let $y_1=(0,1)$, $y_2=(1,0)$, $y_1'=(0,2)$, $y_2'=(2,0)$. Then $\downarrow y_1\cup\downarrow y_2$ has, in L, at least the upper covers $\downarrow y_1'\cup\downarrow y_2$ and $\downarrow y_1\cup\downarrow y_2'$, so $f(2)\geqslant 2$. These are also the only upper covers in L that are subsets of Q. Thus there exists $y\in P\backslash Q$ such that $\mathrm{LC}(y)\subseteq\downarrow y_1\cup\downarrow y_2$; hence $\mathrm{LC}(y)=\{y_1,y_2\}$.

Suppose for a contradiction that $f(2)>3$. Then there exists $z\in P\backslash Q$ distinct from y such that $\mathrm{LC}(z)=\{y_1,y_2\}$ (Fig. 5.1).

Thus $\downarrow y$ has *more* than $f(1)=2$ upper covers in L, namely, $\downarrow y_1'\cup\downarrow y$, $\downarrow y_2'\cup\downarrow y$, and $\downarrow y\cup\downarrow z$, a contradiction.

Hence $f(2)=3$. ∎

For Lemmas 5.3 and 5.4, let the following situation hold: Fix $a\in\mathbb{N}$ and $b\in\mathbb{N}_0$. Let R be a down-set of P isomorphic to the following subposet of $\mathbb{N}_0\times\mathbb{N}_0$: $\{(m,n)\in\mathbb{N}_0\times\mathbb{N}_0\mid m<a\}\cup\{(a,n)\in\mathbb{N}_0\times\mathbb{N}_0\mid n\leqslant b\}\cup\{(m,n)\in\mathbb{N}_0\times\mathbb{N}_0\mid n=0\}$ (Fig. 5.2).

LEMMA 5.3. *Let $I\in\mathcal{O}_{\mathrm{f}}(P)$ be a subset of R that does not contain $(a+1,0)$ or both (a,b) and $(a-1,b+1)$. Let $k\in\mathbb{N}_0$.*

If I has k lower covers in L, then I has $k+1$ upper covers in L that are subsets of R.

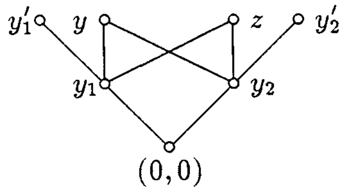

FIG. 5.1. An impossible scenario ($f(0)=1$, $f(1)=2$, $f(2)>3$).

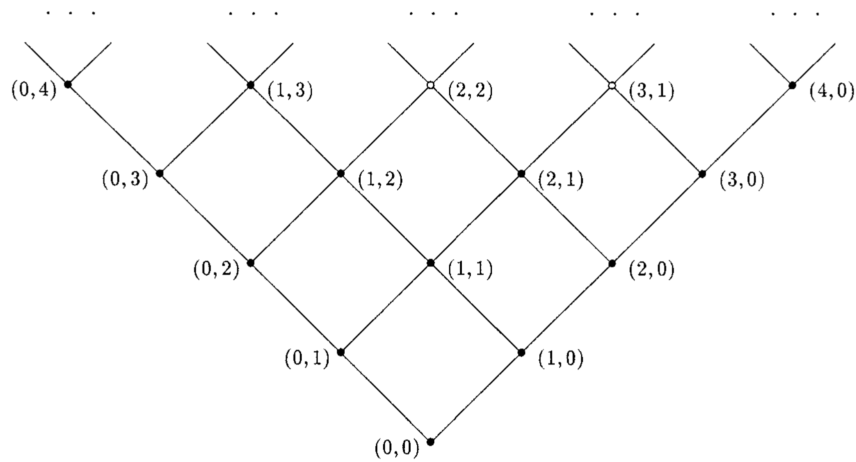

FIG. 5.2. The subset R ($a = 2$, $b = 1$).

Demonstration. A finite down-set I of R can be represented by zig-zags (Fig. 5.3).

Provided none of the "valleys" is $\{(a, b), (a-1, b), (a-1, b+1)\}$, I has upper covers in L obtained by squaring out the valleys and incrementing the ends (Figs. 5.4a–c and 5.5a–d). ∎

LEMMA 5.4. *There exists $y \in P$ such that $\downarrow y = \{y\} \cup \downarrow(a-1, b+1) \cup \downarrow(a, b)$. Label this element $(a, b+1)$. Then $R \cup \{y\}$ is a down-set of P isomorphic to the corresponding subposet of $\mathbb{N}_0 \times \mathbb{N}_0$.*

Proof. By Lemma 5.2(2), exactly $f(2) = 3$ elements of

$$P \backslash (\downarrow(a-1, b+1) \cup \downarrow(a, b))$$

are such that their lower covers all lie in $\downarrow(a-1, b+1) \cup \downarrow(a, b)$. Exactly two of these, $(a+1, 0)$ and $(0, b+2)$, lie in R. Let $y \in P \backslash R$ be the third. Let $I = R \cap \downarrow y \in \mathcal{O}_f(P)$.

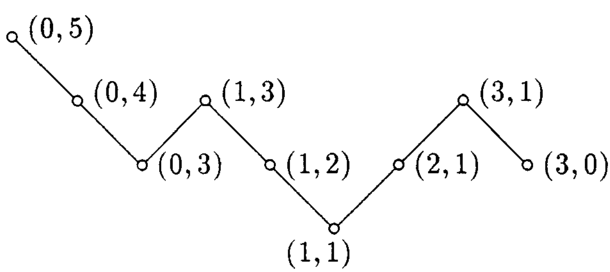

FIG. 5.3. A representation of the down-set I.

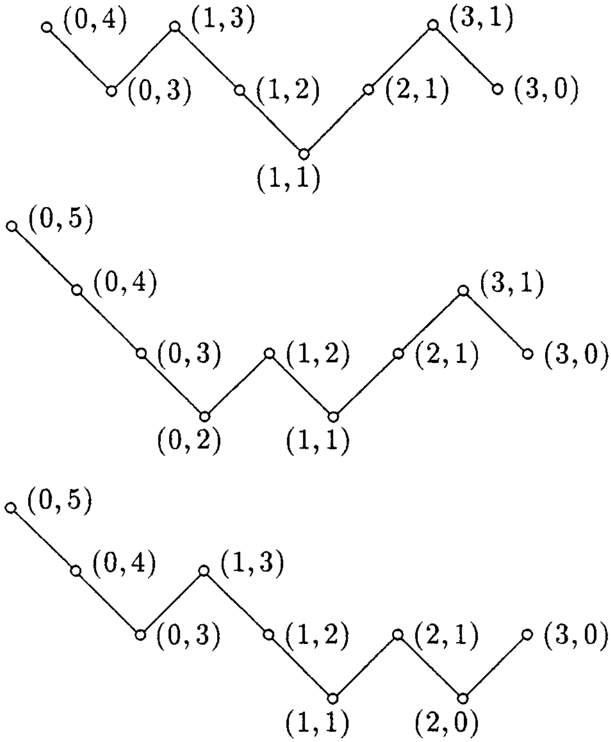

FIG. 5.4. (a)–(c) A representation of a lower cover of I in L.

Assume for a contradiction that I does not contain both of the elements (a, b) and $(a-1, b+1)$. By Lemma 5.3, I has $k+1$ upper covers in L that are subsets of R, for some $k \geqslant 1$. Clearly, for each of these upper covers A, $A \cup \{y\}$ is a different upper cover of $\downarrow y$ in L. Since $f(1)=2$, $k+1 \leqslant 2$, so that $k=1$; i.e., $I = \downarrow z$ for some $z \in R$. Then I has *more* than $f(1)=2$ upper covers in L, two that are subsets of R, and $\downarrow y$, a contradiction.

Thus $\downarrow y$ contains both (a, b) and $(a-1, b+1)$, so y covers both in P and $\mathrm{LC}(y) = \{(a-1, b+1), (a, b)\}$. ∎

Corollary 5.5. *There is a down-set of P isomorphic to $\mathbb{N}_0 \times \mathbb{N}_0$.*

Proof. Lemma 5.1 provides us with the base of an induction ($a=1$, $b=0$ in the set-up preceding Lemma 5.3). Use Lemma 5.4. ∎

Corollary 5.6. *The following hold*:

(1) $P \cong \mathbb{N}_0 \times \mathbb{N}_0$;

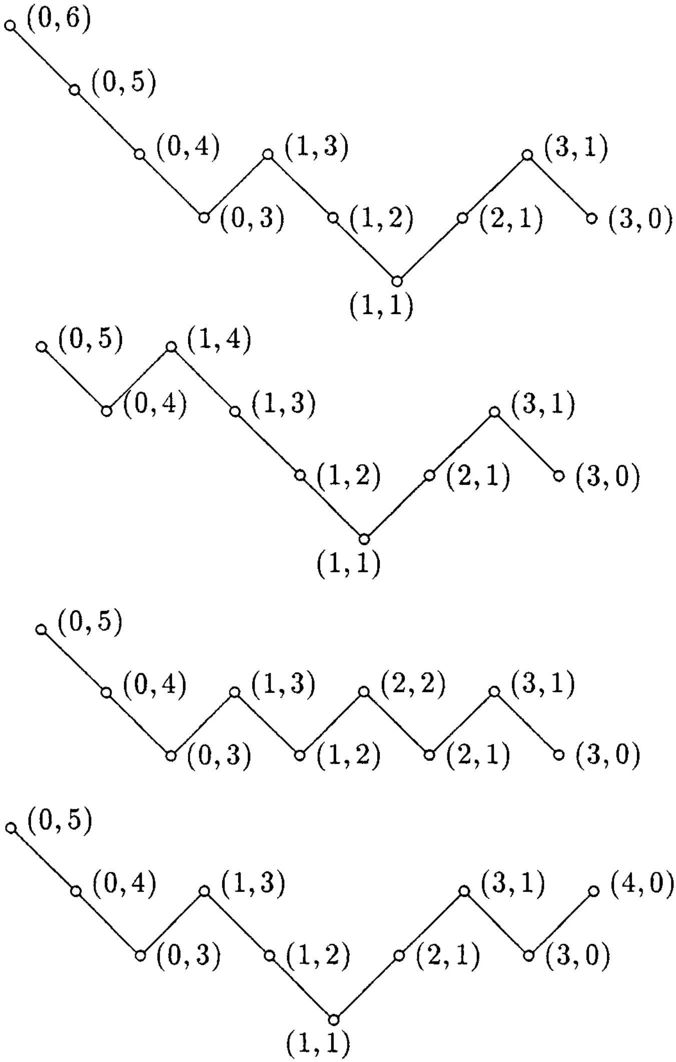

FIG. 5.5. (a)–(d) A representation of an upper cover of I in L.

(2) $L \cong Y$;

(3) $f(n) = n + 1$ *for all* $n \in \mathbb{N}_0$.

Proof. Let S be the down-set of Corollary 5.5, and assume for a contradiction that $P \backslash S \neq \varnothing$. Let $y \in P \backslash S$ be minimal. By the same argument as in the proof of Lemma 5.4 we get a contradiction.

The rest follows by definition and Lemma 4.8. ∎

6. $f(0) \geqslant 2$; $f(1) = f(0) + 1$

LEMMA 6.1. *Let* $Q = Q_1 = \{p \in P \mid \downarrow p \cap \{x_2, ..., x_{f(0)}\} = \varnothing\}$. *Then*:

(1) Q *is a down-set of* P;

(2) $\mathcal{O}_{\mathrm{f}}(Q)$ *is a finitary distributive lattice with cover function*

$$g(n) = f(n) - f(0) + 1 \qquad (n \in \mathbb{N}_0).$$

Proof. Clearly $\mathcal{O}_{\mathrm{f}}(Q)$ is a down-set of $\mathcal{O}_{\mathrm{f}}(P)$. Let $I \in \mathcal{O}_{\mathrm{f}}(Q)$ have n lower covers in $\mathcal{O}_{\mathrm{f}}(Q)$ (and hence in L) for some $n \in \mathbb{N}_0$. Note that $I \cup \{x_i\}$ $(2 \leqslant i \leqslant f(0))$ are upper covers of I in L, so $f(n) \geqslant f(0) - 1$.

The other $f(n) - (f(0) - 1)$ upper covers of I in L must actually be subsets of Q. That is, I has $f(n) - f(0) + 1$ upper covers in the lattice $\mathcal{O}_{\mathrm{f}}(Q)$. ▮

LEMMA 6.2. *The inequality* $f(2) \geqslant f(0) + 2$ *holds*.

Proof. Let $q = x_1$ in Lemma 4.4; then $A = \{x_2, ..., x_{f(0)}\}$, so x_1 has exactly 2 irreducible upper covers in P, y_1 and y_1'. Similarly, x_2 has 2 irreducible upper covers, y_2 and y_2'.

Hence $\{x_1, x_2\}$ has, in L, at least the upper covers $\{x_1, x_2, x_i\}$ $(3 \leqslant i \leqslant f(0))$, $\downarrow y_1 \cup \{x_2\}$, $\downarrow y_1' \cup \{x_2\}$, $\{x_1\} \cup \downarrow y_2$, and $\{x_1\} \cup \downarrow y_2'$. ▮

COROLLARY 6.3. *The poset* Q *of Lemma* 6.1 *is isomorphic to* $\mathbb{N}_0 \times \mathbb{N}_0$.

Proof. By Lemmas 6.1(2) and 6.2, $g(0) = 1$, $g(1) = 2$, and $g(2) \geqslant 3$. By Corollary 5.6(1), $Q \cong \mathbb{N}_0 \times \mathbb{N}_0$. ▮

LEMMA 6.4. *Define* $Q_1, ..., Q_{f(0)}$ *as in Lemma* 6.1. *Then* $P = \sum_{i=1}^{f(0)} Q_i$.

Proof. Let $S = \bigcup_{i=1}^{f(0)} Q_i$. Clearly $S = \sum_{i=1}^{f(0)} Q_i$.

Suppose, for a contradiction, that $P \backslash S \neq \varnothing$; choose $y \in P \backslash S$ minimal. Let $I_i = \downarrow y \cap Q_i$ $(i = 1, ..., f(0))$. There exist distinct $j, k \in \{1, ..., f(0)\}$ such that $I_j, I_k \neq \varnothing$.

By Corollary 6.3, for $1 \leqslant i \leqslant f(0)$, $Q_i \cong \mathbb{N}_0 \times \mathbb{N}_0$, and, by Lemma 4.8, I_i has at least 1 upper cover in L that is a subset of Q_i (at least 2 if $i \in \{j, k\}$). Hence, $I = \bigcup_{i=1}^{f(0)} I_i$ has at least $f(0) - 2 + 4 = f(0) + 2$ upper covers in L that are subsets of S, so $\downarrow y$ has *more* than $f(1) = f(0) + 1$ upper covers in L, a contradiction. ▮

COROLLARY 6.5. *The following hold*:

(1) $P \cong \sum_{i=1}^{f(0)} (\mathbb{N}_0 \times \mathbb{N}_0)$;

(2) $L \cong Y^{f(0)}$;

(3) $f(n) = n + f(0)$ *for all* $n \in \mathbb{N}_0$.

Proof. Statements (1) and (2) follow from Corollary 6.3 and Lemma 6.4. Let Q be the component containing x_1. For $n \in \mathbb{N}_0$, let $I \in \mathcal{O}_f(Q) \subseteq \mathcal{O}_f(P)$ have n lower covers in $\mathcal{O}_f(Q)$, hence in L. (Such an I exists.) By Lemma 4.8, I has exactly $n+1$ upper covers in L that are subsets of Q, and also the covers $I \cup \{x_i\}$ $(2 \leqslant i \leqslant f(0))$, for a total of $(n+1)+(f(0)-1)=f(0)+n$. ∎

7. $f(0) \geqslant 3$; $f(1)=f(0)$

In this section, let Q be the subset of Corollary 4.6; for $1 \leqslant i \leqslant f(0)$, let x_i' be the irreducible upper cover of x_i.

LEMMA 7.1. *We have* $f(2)=f(0)$.

Proof. Now $\{x_1, x_2\}$ has at least $f(0)$ upper covers in L, namely, $\{x_1', x_2\}$, $\{x_1, x_2'\}$, and $\{x_1, x_2, x_i\}$ $(3 \leqslant i \leqslant f(0))$. Thus $f(2) \geqslant f(0)$.

Assume for a contradiction that $f(2) > f(0)$. Then there are exactly $f(2)-f(0)$ elements $z \in P$ such that $\mathrm{LC}(z)=\{x_1, x_2\}$, $y=y_1, ..., y_{f(2)-f(0)}$ (Fig. 7.1).

In fact, $f(2)-f(0)=1$. [For, otherwise, $\downarrow y_1$ would have *more* than $f(1)$ upper covers in L, namely, $\downarrow x_1' \cup \downarrow y_1$, $x_2' \cup \downarrow y_1$, $\downarrow y_1 \cup \downarrow y_2$, and $\downarrow y_1 \cup \{x_i\}$ $(3 \leqslant i \leqslant f(0))$.]

Let $y' \in P$ be the unique element such that $\mathrm{LC}(y')=\{x_1, x_3\}$ and similarly choose y'' for $\{x_2, x_3\}$ (Fig. 7.2).

Now $\{x_3\} \cup \downarrow y$ has *more* than $f(2)=f(0)+1$ upper covers in L, namely, $\{x_1', x_3\} \cup \downarrow y$, $\{x_2', x_3\} \cup \downarrow y$, $\downarrow x_3' \cup \downarrow y$, $\downarrow y \cup \downarrow y'$, $\downarrow y \cup \downarrow y''$, and $\{x_3, x_i\} \cup \downarrow y$ $(4 \leqslant i \leqslant f(0))$, a contradiction. ∎

LEMMA 7.2. *For* $0 \leqslant n \leqslant f(0)$, $f(n)=f(0)$.

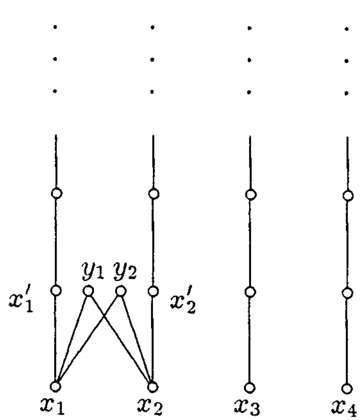

FIG. 7.1. An impossible scenario ($f(0)=4$, $f(1)=4$, $f(2)=6$).

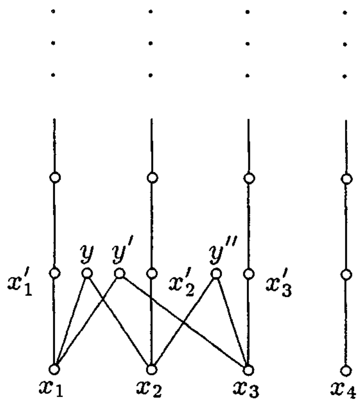

FIG. 7.2. An impossible scenario ($f(0)=4, f(1)=4, f(2)=5$).

Proof. The case $0 \leqslant n \leqslant 2$ is Lemma 7.1. Now assume $3 \leqslant n \leqslant f(0)$ and that $f(k)=f(0)$ for $0 \leqslant k < n$.

Note that no element of P is above exactly k of the elements $x_1, ..., x_{f(0)}$ if $2 \leqslant k < n$. The only elements above exactly one of $x_1, ..., x_{f(0)}$ are in Q. (We are using the irreducibility properties in Corollary 4.6.)

Let $I = \{x_1, ..., x_n\}$. It has exactly n lower covers in L; it also has exactly $f(0)$ upper covers in L that are subsets of Q, namely, $I \cup \{x_i'\}$ $(1 \leqslant i \leqslant n)$ and $I \cup \{x_i\}$ $(n < i \leqslant f(0))$. Hence $f(n) \geqslant f(0)$.

Suppose for a contradiction that $f(n) > f(0)$. Then there exist $f(n) - f(0)$ elements w such that $\mathrm{LC}(w) = \{x_1, ..., x_n\}$, $y = y_1, ..., y_{f(n)-f(0)}$ (Fig. 7.3).

(In fact, $f(n) - f(0) = 1$, or else $\downarrow y$ would have *more* than $f(1) = f(0)$ upper covers in L.)

There are exactly $f(0)$ upper covers of $\downarrow y \cup \{x_2', ..., x_n'\}$ that are subsets of Q; hence there exists $z_1 \in P \backslash (\downarrow y \cup \{x_2', ..., x_n'\} \cup Q)$ such that

$$\mathrm{LC}(z_1) \subseteq \downarrow y \cup \{x_2', ..., x_n'\}$$

(Fig. 7.4).

We must have $y \lessdot z_1$, or else $\downarrow x_1' \cup \cdots \cup \downarrow x_n'$ would have *more* than $f(n) = f(0) + 1$ upper covers in L, $f(0)$ being subsets of Q, and then

$$\downarrow x_1' \cup \cdots \cup \downarrow x_n' \cup \{y\} \qquad \text{and} \qquad \downarrow x_1' \cup \cdots \cup \downarrow x_n' \cup \{z_1\}$$

(Fig. 7.5).

Indeed, $x_n' \in \mathrm{LC}(z_1)$, for, otherwise, $\downarrow y \cup \downarrow x_2' \cup \cdots \cup \downarrow x_{n-1}'$ would have *more* than $f(n-1) = f(0)$ upper covers in L. Similarly, $x_i' \lessdot z_1$ $(2 \leqslant i \leqslant n)$.

Now choose $z_2 \in P \backslash (\downarrow y \cup \{x_1', x_3', ..., x_n'\} \cup Q)$ such that

$$\mathrm{LC}(z_2) \subseteq \downarrow y \cup \{x_1', x_3', ..., x_n'\}$$

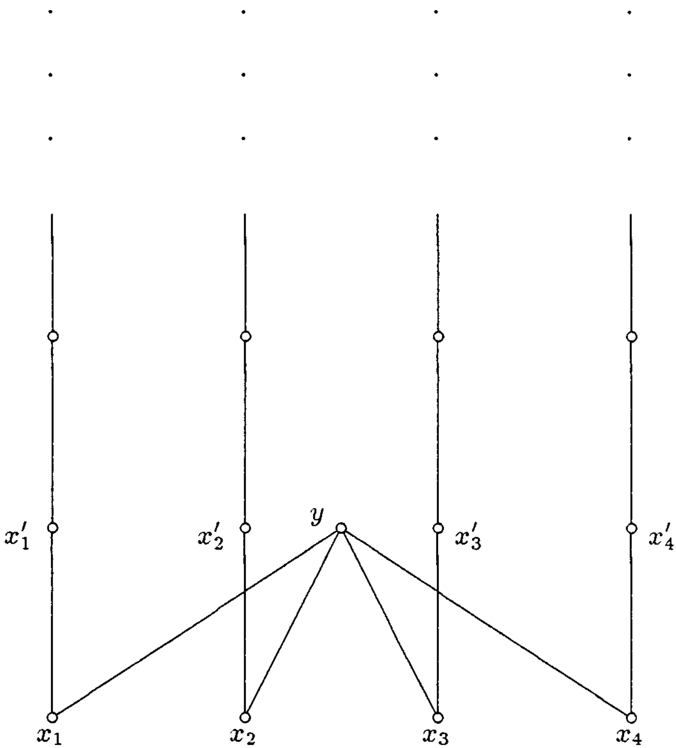

FIG. 7.3. An impossible scenario ($f(0)=4$, $f(1)=4$, $n=4$, $f(n)>4$).

and $z_3 \in P \backslash (Q \cup \downarrow y \cup \{x'_1, ..., x'_n\} \backslash \{x'_3\})$ such that

$$\mathrm{LC}(z_3) \subseteq \downarrow y \cup (\{x'_1, ..., x'_n\} \backslash \{x'_3\})$$

(Fig. 7.6).

Of course, all three of z_1, z_2, z_3 are distinct, as $x'_2 < z_1$ but $x'_2 \nless z_2$.

Note that $\downarrow z_1 \cup \downarrow z_2$ has *more* than $f(2)=f(0)$ upper covers in L, $f(0)$ of them involving adding elements from Q, and then $\downarrow z_1 \cup \downarrow z_2 \cup \downarrow z_3$, a contradiction. ▮

LEMMA 7.3. *No element of L has more than $f(0)$ lower covers, and $P=Q$.*

Proof. Assume, for a contradiction, that $P \backslash Q \neq \varnothing$. Choose $p \in P \backslash Q$ minimal. Then $I = \downarrow p \cap Q \in \mathcal{O}_f(P)$ has *more* than $f(0)$ upper covers in L, for it has $f(0)$ upper covers in L that are subsets of Q, and also $\downarrow p$, a contradiction. ▮

The following is clear:

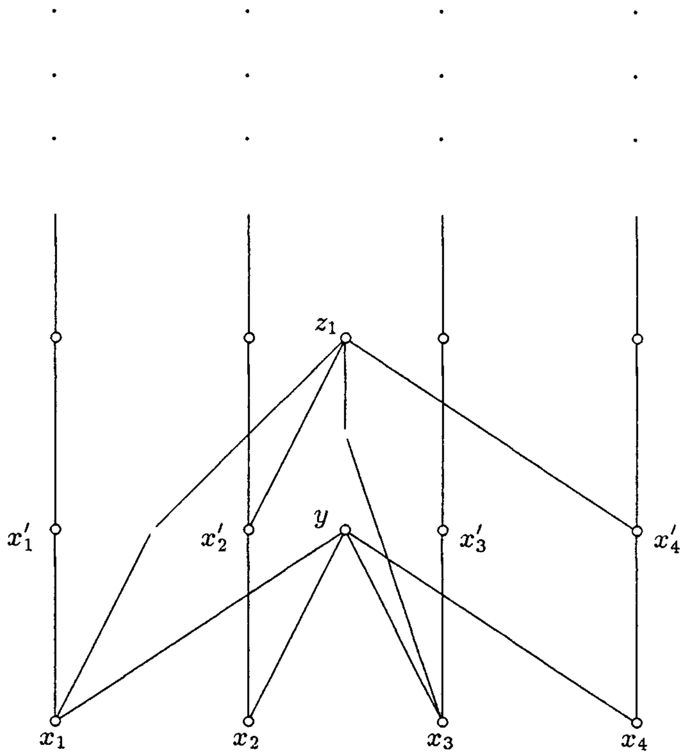

FIG. 7.4. An impossible scenario ($f(0)=4$, $f(1)=4$, $n=4$, $f(n)>4$).

COROLLARY 7.4. *The following hold*:

(1) $P \cong \sum_{i=1}^{f(0)} \mathbb{N}$;
(2) $L \cong \mathbb{N}_0^{f(0)}$;
(3) $f(n) = f(0)$ *if* $0 \leqslant n \leqslant f(0)$. ∎

8. $f(0)=2$; $f(1)=f(0)$; $f(2)=f(0)$

COROLLARY 8.1. *The following hold*:

(1) $P \cong \mathbb{N} + \mathbb{N}$;

(2) $L \cong \mathbb{N}_0^2$;

(3) $f(n) = 2$ *if* $0 \leqslant n \leqslant 2$.

Proof. It suffices to show that the poset Q of Corollary 4.6 is all of P. Use the argument of Lemma 7.3. ∎

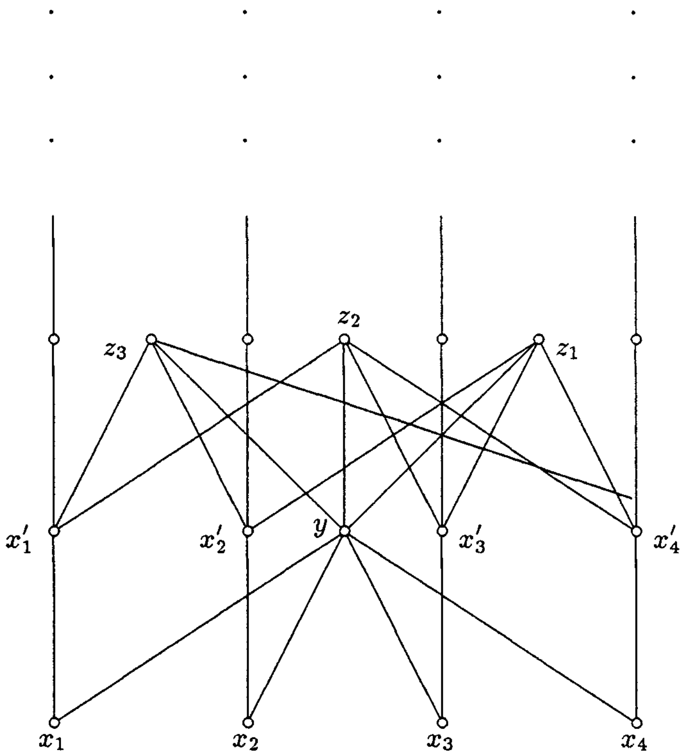

FIG. 7.6. An impossible scenario ($f(0)=4$, $f(1)=4$, $n=4$, $f(n)>4$).

Proof. This is obvious. ▮

COROLLARY 10.2. *If $f(1)=2$ and $f(2)=f(1)$, then the following hold*:

(1) $P \cong \mathbf{1} \oplus (\mathbb{N}+\mathbb{N})$;
(2) $L \cong \mathbf{1} \oplus \mathbb{N}_0^2$;
(3)

$$f(n)=\begin{cases}1 & \text{if } \; n=0,\\ 2 & \text{if } \; 1 \leqslant n \leqslant 2.\end{cases}$$

Proof. See Corollary 8.1 and Lemma 4.1. ▮

COROLLARY 10.3. *If $f(1)=2$ and $f(2) \neq f(1)$, then $L \cong Y$.*

Proof. See Corollary 5.6. ▮

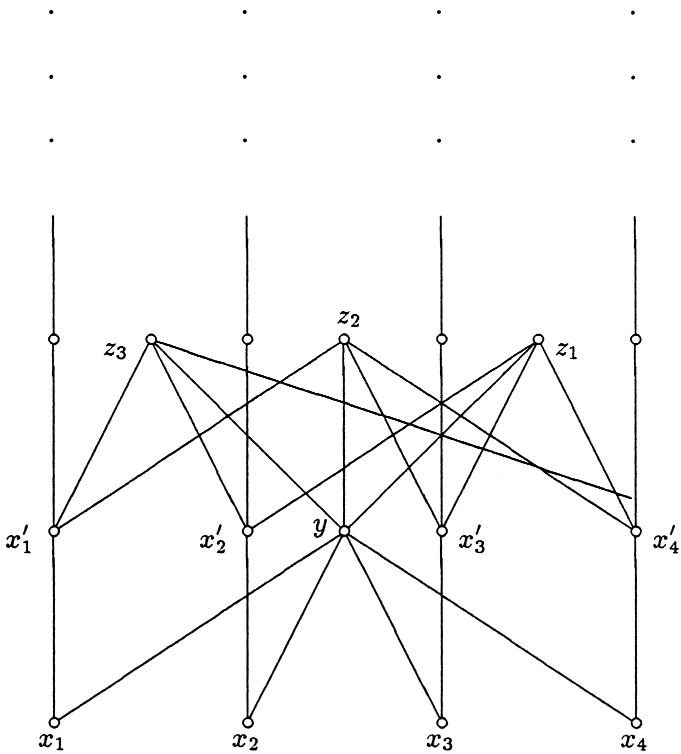

FIG. 7.6. An impossible scenario ($f(0)=4$, $f(1)=4$, $n=4$, $f(n)>4$).

Proof. This is obvious. ∎

COROLLARY 10.2. *If $f(1)=2$ and $f(2)=f(1)$, then the following hold*:

(1) $P \cong \mathbf{1} \oplus (\mathbb{N}+\mathbb{N})$;
(2) $L \cong \mathbf{1} \oplus \mathbb{N}_0^2$;
(3)

$$f(n)=\begin{cases}1 & \textit{if} \quad n=0,\\ 2 & \textit{if} \quad 1 \leqslant n \leqslant 2.\end{cases}$$

Proof. See Corollary 8.1 and Lemma 4.1. ∎

COROLLARY 10.3. *If $f(1)=2$ and $f(2) \neq f(1)$, then $L \cong Y$.*

Proof. See Corollary 5.6. ∎

COROLLARY 10.4. *If* $f(1)\geqslant 3$, *then the following hold*:

(1) $P\cong \mathbf{1}\oplus\sum_{i=1}^{f(1)}\mathbb{N}$;

(2) $L\cong \mathbf{1}\oplus\mathbb{N}_0^{f(1)}$;

(3)

$$f(n)=\begin{cases}1 & \text{if } n=0,\\ f(1) & \text{if } 1\leqslant n\leqslant f(1).\end{cases}$$

Proof. Use Lemma 4.1 and Corollary 7.4. ▮

11. *The Characterization of Non-decreasing Cover Functions*

THEOREM 11.1. *Let L be a finitary distributive lattice with non-decreasing cover function* $f\colon \mathbb{N}_0\to\mathbb{N}_0$. *Then one of the following holds*:

(1) *for some* $k\geqslant 1$, $L\cong\mathbb{N}_0^k$; *and for all* $n\in\mathbb{N}_0$,

$$f(n)=\begin{cases}k & \text{if } 0\leqslant n\leqslant k,\\ * & \text{otherwise};\end{cases}$$

(2) *for some* $k\geqslant 1$, $L\cong Y^k$; *and for all* $n\in\mathbb{N}_0$, $f(n)=n+k$;

(3) *for some* $k\geqslant 2$, $L\cong\mathbf{1}\oplus\mathbb{N}_0^k$; *and for all* $n\in\mathbb{N}_0$,

$$f(n)=\begin{cases}1 & \text{if } n=0,\\ k & \text{if } 1\leqslant n\leqslant k,\\ * & \text{otherwise};\end{cases}$$

(4) $L\cong Y\backslash\{0_Y\}$; *and for all* $n\in\mathbb{N}_0$,

$$f(n)=\begin{cases}2 & \text{if } n=0,\\ n+1 & \text{if } n\geqslant 1;\end{cases}$$

(5) $L\cong\mathbf{1}$; *and for all* $n\in\mathbb{N}_0$,

$$f(n)=\begin{cases}0 & \text{if } n=0,\\ * & \text{otherwise}.\end{cases}$$

Moreover, the functions listed are cover functions for the corresponding finitary distributive lattices.

Proof. If $f(0)=0$, we have (5).

If $f(0)=1$, we have (1) (Corollary 10.1), (3) (Corollary 10.2), (2) (Corollary 10.3), or (3) (Corollary 10.4).

If $f(0) \geqslant 2$ and $f(1) \neq f(0)$, by Lemma 4.7 we have (2) (Corollary 6.5). Else, if $f(0) = 2$, we have (1) (Corollary 8.1) or (4) (Corollary 9.1). If $f(0) \geqslant 3$ and $f(1) = f(0)$, we have (1) (Corollary 7.4). ∎

Thus the 1975 conjecture of Stanley is settled.

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