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# Linear extensions of ranked posets, enumerated by descents. A problem of Stanley from the 1981 Banff Conference on Ordered Sets

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**Abstract**

Let $P$ be a naturally labelled, ranked (graded) poset of rank $r$ and cardinality $n$. Let $H_k$ be the set of linear extensions of $P$ with $k$ descents. An explicit bijection between $H_k$ and $H_{n-1-r-k}$ is constructed using the involution principle ($0 \leqslant k \leqslant n-1-r$). A problem of Richard P. Stanley from 1981 is thereby solved.


*Keywords:* (Partially) Ordered set; Linear extension; Natural labelling; Descent; $h$-vector; $P$-partition; Eulerian numbers

## 1. The five of hearts

Major Percy MacMahon, that great British combinatorialist of the turn of the last century, proved the following result in his classic *Combinatory Analysis* [11, Section IV, Chapter V, Sect. 179–180, pp. 212–213].

Take $m$ different numbers (say, the integers 1 through $m$), each number repeated $r+1$ times, so that there are $n = m(r+1)$ numbers in all. Consider all possible ways of listing these $n$ numbers in a row; if $r = 0$, we are just listing all possible permutations of $m$ objects. (Knuth uses the analogy of shuffling a deck of cards, where suit is ignored: in this case,

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Table 1.1
MacMahon's theorem for $m = 2$ and $r = 2$

| $H_0$ | $H_1$ | $H_2$ | $H_3$ |
|---|---|---|---|
| 111222 | 112122 | 221211 | 212121 |
| | 112212 | 212211 | |
| | 112221 | 212112 | |
| | 121122 | 221121 | |
| | 122112 | 211221 | |
| | 122211 | 211212 | |
| | 211122 | 122121 | |
| | 221112 | 121221 | |
| | 222111 | 121212 | |

$m = 13$ and $r = 3$ [9, p. 43].) For each listing, count the number of "descents," the number of places where a bigger number *immediately* precedes a smaller number.

For instance, if $m = 2$ and $r = 2$, there are 20 possibilities (see Table 1.1).

Let $H_k$ be the set of sequences with exactly $k$ descents and let $h_k = |H_k|$, the number of such sequences. Table 1.1 shows that $h_0 = h_3$ and $h_1 = h_2$, that is, *the "$h$-vector" $(h_0, h_1, h_2, h_3)$ is symmetric.* MacMahon proved in general that

$$h_k = h_{n-1-r-k} \quad (0 \leqslant k \leqslant n - 1 - r).$$

MacMahon's proof used generating functions: he did not directly establish a one-to-one correspondence between $H_k$ and $H_{n-1-r-k}$. Indeed, writes Knuth, "No very simple correspondence is evident" except in trivial cases. (Knuth then goes on to establish such a bijection—an algorithm, really—using Foata's idea of expressing multipermutations as products of cycles [9, pp. 24–29, 43–44].)

A curious result, to be sure—"quite surprising," Knuth says—but does it tell us anything about anything else? That is, does it *generalize*?

Generalize *how*? one might ask. To answer that question, we must translate MacMahon's result into the language of ordered sets.

The plan of this paper is as follows. All definitions are contained in Section 5. In Section 2 we reveal Stanley's generalization of MacMahon's theorem. In Section 3 we state Stanley's problem. In Section 4 we mention related results from the literature. In Section 5 we solve Stanley's problem. In Section 6 we illustrate our solution with an example. In Section 7 we describe avenues for further research. In the appendix we illustrate posets described in the main body of this work. In Section 1 we give a plan of the paper...

## 2. Everything I needed to know I learned from the four-element posets

Instead of multipermutations of words with the letters

$$1, \ldots, 1, 2, \ldots, 2, 3, \ldots, 3, \ldots, m, \ldots, m,$$

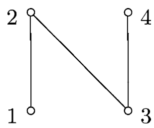
let us use *permutations* of the set $1, 2, \ldots, n$. The translation is illustrated in Fig. 2.1.

Fig. 2.1. Translating MacMahon’s theorem to the language of posets.

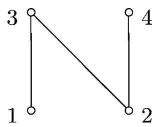
In any shuffling, such as 211212, replace the first 1 by 1, replace the second 1 by 2, replace the third 1 by 3, ..., replace the $(r+1)$st 1 by $r+1$; replace the first 2 by $r+2$, etc.; thus 211212 becomes 412536. Of course, we cannot get *any* permutation on $n$ letters this way; we only get a permutation if, whenever $\rho < \rho'$ ($\rho, \rho'$ elements of the poset on the right of Fig. 2.1), the numerical label of $\rho$ appears to the left of the label for $\rho'$. Such a permutation is called a *linear extension* of the poset. (It is clear that a shuffling has $k$ descents if and only if its translate does.)

A labelling of the elements of a finite poset with the letters $1, \ldots, n$ so that $123 \cdots n$ is a linear extension is called a *natural labelling*. Given a finite poset $P$ with a natural labelling, we can define $H_k$ to be the set of linear extensions (permutations compatible with the order on $P$) with $k$ descents, and set $h_k = |H_k|$ as before.

Figure 2.2 shows a four-element poset with an unnatural labelling (illegal in some states); that same poset with *two* different natural labellings; and their corresponding sets of

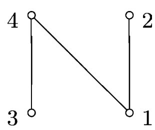


(a) The poset $N$ with an unnatural labelling.



| | |
|---|---|
| 1234 | 0 descents |
| 2134 | 1 descent |
| 1243 | 1 descent |
| 2413 | 1 descent |
| 2143 | 2 descents |

(b) The naturally labelled poset $N$ has $h_0 = 1$, $h_1 = 3$, $h_2 = 1$.



| | |
|---|---|
| 1234 | 0 descents |
| 1324 | 1 descent |
| 1342 | 1 descent |
| 3124 | 1 descent |
| 3142 | 2 descents |

(c) The poset $N$ with another natural labelling.

Fig. 2.2.

linear extensions. Note that, while the *set* $H_k$ depends on the natural labelling, the number $h_k$ does not. (See, for instance, [20, Theorem 3.12.1].)

Figure A.1 in the appendix lists some other posets along with their $h$-vectors $(h_0, h_1, h_2, \ldots)$. (Strictly speaking, these are the $h$-vectors of the order complexes of the lattice of down-sets of these posets; see [1, Section 5.1] and [23, Section 8.3].)

Note that when $P$ is an antichain, we get the classical *eulerian numbers*, and also note that a standard Young tableau is just a linear extension of a certain poset [14,16, pp. 43–44].

To illustrate, in Table A.1 we list all 24 permutations on four letters (Fig. A.1(f)), and mark those that are not linear extensions of the naturally labelled poset of Fig. A.1(g). Table A.2 lists the linear extensions of the naturally labelled poset of Fig. A.1(k); Table A.3 the linear extensions of the posets of Figs. 2.1 and A.1(l); and Table A.4 the linear extensions of the poset of Fig. A.1(m).

We note that, for each poset $P$, the index of the largest non-zero $h_k$ is $k = n - 1 - r$, where $n = |P|$ and $r + 1$ is the cardinality of the longest chain (totally ordered subset). (See the easy Lemma 5.1 or [16, Theorem 16.1].) Moreover, the $h$-vector $(h_0, \ldots, h_{n-1-r})$ is symmetric just when $P$ is *ranked* (*graded*), that is, when every maximal chain has the same cardinality. This is the content of Stanley's generalization of MacMahon's theorem.

**Theorem** (Stanley). *Let $P$ be a finite naturally labelled poset. Let $\mathcal{L}(P)$ be the set of linear extensions of $P$, and, for every $\pi \in \mathcal{L}(P)$, let $d(\pi)$ be the number of descents of $\pi$. Let $M$ be* $\max\{d(\pi) \mid \pi \in \mathcal{L}(P)\}$. *Then the following are equivalent*:

(i) $h_k = h_{M-k}$ *for* $0 \leqslant k \leqslant M$,
(ii) $P$ *is ranked.*

## 3. The statement of Stanley's problem from the 1981 Banff Conference on Ordered Sets

At the 1981 Banff Conference on Ordered Sets [13, p. 807], Stanley said, "About ten years ago I proved (the above result)." He went on to pose the following

**Problem** (Stanley, 1981). Find a combinatorial proof of this theorem. More precisely, when (ii) holds describe explicitly a bijection $f : \mathcal{L}(P) \overset{\simeq}{\to} \mathcal{L}(P)$ such that $d(\pi) = M - d(f(\pi))$ for all $\pi \in \mathcal{L}(P)$.

(Stanley added, "It would even be interesting to do this for the case $P \cong \mathbf{r} \times \mathbf{s}$ (the product of an $r$-element chain and an $s$-element chain).")

We solve Stanley's problem by constructing a bijection

$$\Phi_{k,k} : H_k \to H_{n-1-r-k}$$

for $k \in \{0, \ldots, M = n - 1 - r\}$ where $|P| = n$ and every maximal chain has $r + 1$ elements (Theorem 5.8).

## 4. Background and previous results

Basic references on posets are [2] and [20, Chapter 3]. We will not assume a poset is ranked without explicitly saying so. Because of the vast literature on $f$-vectors and $h$-vectors of polytopes and posets, permutation statistics, etc., we limit ourselves to recalling results most directly related to the present work, results concerning inequalities for $h$-vectors (which are also called $w$-vectors). Relevant papers (albeit not essential for understanding this work) include the very interesting [15], as well as [6,7] (see its Corollary 2.6) and [8] (see its Theorem 2.4), where Hibi shows, invoking a commutative algebra result [19, Theorem 2.1], that

$$h_0 + h_1 + \cdots + h_k \leqslant h_M + h_{M-1} + \cdots + h_{M-k} \quad \left(0 \leqslant k \leqslant \left\lfloor \frac{M}{2} \right\rfloor\right).$$

He states the following

**Conjecture** (Hibi, 1991). *For* $0 \leqslant k \leqslant \lfloor \frac{M}{2} \rfloor$,

$$h_k \leqslant h_{M-k} \quad \textit{and} \quad h_0 \leqslant h_1 \leqslant \cdots \leqslant h_{\lfloor \frac{M}{2} \rfloor}.$$

In the proof of [5, Theorem 1.2], Gasharov provides a bijection from $H_k$ to $H_{n-1-r-k}$ when the rank $r$ of the poset is 1 or 2, where we use the definition of rank that says that an antichain has rank 0. (He also proves that the $h$-vector is unimodal.) He writes, “The proof that we provide for Theorem 1.2 can be considered combinatorial, although we do not explicitly exhibit the necessary injections as this would be rather cumbersome.”

Reiner and Welker [12] prove that, when $P$ is ranked, the $h$-vector is symmetric and unimodal by invoking the (decidedly non-trivial) $g$-Theorem for simplicial polytopes [18]; but this is not a combinatorial proof.

Fix a poset $P$ of cardinality $n$. Let $\Omega(P, m)$ denote the number of order-preserving maps from $P$ to an $m$-element chain and let $\overline{\Omega}(P, m)$ denote the number of *strictly* order-preserving maps. These are polynomials in $m$ (the *order polynomial* and the *strict order polynomial*, respectively). Stanley’s reciprocity theorem for order polynomials ([17, Proposition 2.1], [20, Corollary 4.5.15]) states that

$$\overline{\Omega}(P, m) = (-1)^n \Omega(P, -m).$$

(Kreweras concedes being initially unaware of Stanley’s results, but his exposition is still interesting [10].) Though partially hidden, our Proposition 5.5 really amounts to analyzing the reciprocity theorem and its ingredients from Stanley’s theory of $P$-partitions and considerations like those in [16, Section 18]. (We obtain the final bijection using the involution principle.)

Thus we see that Stanley could have solved Stanley’s problem by reading Stanley.

## 5. The solution to Stanley's problem

We will use the following notation and definitions throughout this section and the next.

All numbers will be non-negative integers. For $n \geqslant 0$, let $[n] := \{1, \dots, n\}$ and let $[n]_0 := \{0, \dots, n\}$. (If we have an expression like $\{1, \dots, n\}$ where $n = 0$, then we mean the empty set.) Let $|S|$ denote the cardinality of the finite set $S$. If $X$, $Y$, $X'$, and $Y'$ are sets with $X \cap Y = \emptyset$, and if $f : X \to X'$ and $g : Y \to Y'$ are functions, define $f \cup g : X \cup Y \to X' \cup Y'$ to be the function such that, for every $z \in X \cup Y$,

$$(f \cup g)(z) = \begin{cases} f(z) & \text{if } z \in X, \\ g(z) & \text{if } z \in Y. \end{cases}$$

A *multiset* is a family with repetitions (so $\{1, 2, 2, 3\} \neq \{1, 2, 3\}$ as multisets). We define cardinality, union, and complementation for multisets appropriately, so

$$\bigl|\{1, 2, 2, 3\}\bigr| = 4, \quad \{1, 2\} \cup \{2, 3\} = \{1, 2, 2, 3\}, \quad \text{and}$$

$$\{1, 2, 2, 3\} \setminus \{1, 2\} = \{2, 3\}.$$

For $k \geqslant 0$, let $\left(\!\binom{S}{k}\!\right)$ denote the family of cardinality $k$ multisets with elements drawn from the set $S$; if $d_1, \dots, d_k$ are numbers ($k \geqslant 0$), the notation $\{d_1, \dots, d_k\}_{\leqslant}$ for the corresponding multiset indicates that

$$d_1 \leqslant \dots \leqslant d_k.$$

Let $P$ be a finite alphabet (set). If $w$ is a word $\sigma_1 \cdots \sigma_k$ with $k$ letters ($k \geqslant 0$; $\sigma_1, \dots, \sigma_k \in P$), the *length* $|w|$ of $w$ is $k$; we say the letter $\sigma_i$ *appears* in $w$ ($i \in [k]$); and, if $1 \leqslant i < j \leqslant k$, that $\sigma_i$ *appears to the left* of $\sigma_j$ in $w$. If $w_1 = \sigma_1 \cdots \sigma_k$ and $w_2 = \tau_1 \cdots \tau_l$ are words ($k, l \geqslant 0$; $\sigma_1, \dots, \sigma_k, \tau_1, \dots, \tau_l \in P$), then the *concatenation* of $w_1$ and $w_2$, denoted $w_1 w_2$, is the word $\sigma_1 \cdots \sigma_k \tau_1 \cdots \tau_l$.

A non-empty finite poset $P$ is *ranked* of *rank* $r$ if all maximal chains (totally ordered subsets maximal with respect to set-inclusion) have the same cardinality $r + 1$; the *rank* $r(\rho)$ of an element $\rho \in P$ is the rank of the subposet $\{\rho' \in P \mid \rho' \leqslant \rho\}$.

Fix a finite ranked poset $P$ of cardinality $n \geqslant 2$ and rank $r$. Fix an order-preserving bijection from $P$ to the chain $[n]$ and label the elements of $P$ as $\rho_1, \rho_2, \dots, \rho_n$ so that $\rho_i \mapsto i$ ($i \in [n]$). (This is called a *natural labelling*.)

If

$$w = \rho_{i_1} \cdots \rho_{i_k}$$

is a word drawn from the alphabet $P$ (where $k \geqslant 0$; $i_1, \dots, i_k \in [n]$), then we say $w$ is *in increasing order* if $i_1 < \dots < i_k$; and *in decreasing order* if $i_1 > \dots > i_k$.

A *linear extension* of $P$ is a word

$$w = \rho_{i_1} \cdots \rho_{i_n} \quad \bigl(i_1, \dots, i_n \in [n]\bigr)$$

with $n$ distinct letters such that, if $\rho < \rho'$ in $P$, then $\rho$ appears to the left of $\rho'$ in $w$. The *descent set* $D(w)$ of such a linear extension $w$ is the set $\{j \in [n-1] \mid i_j > i_{j+1}\}$ and the *ascent set* $A(w)$ is $\{j \in [n-1] \mid i_j < i_{j+1}\}$; we say $w$ has $k$ *descents* and $l$ *ascents* if $k = |D(w)|$ and $l = |A(w)|$. Let $H_k$ be the set of linear extensions of $P$ with $k$ descents.

For the next three paragraphs, fix $k \in [n-1-r]_0$. If $l \leqslant k$, let

$$\mathcal{D}_{k,l} := \left\{ (w, D) \in H_l \times \left( \binom{[n-1]}{k} \right) \,\middle|\, D(w) \subseteq D \right\}$$

and

$$\mathcal{A}_{k,l} := \left\{ (v, A) \in H_{n-1-r-l} \times \left( \binom{[n-1]}{r+k} \right) \,\middle|\, A(v) \subseteq A \right\}.$$

Let $\mathcal{D}_k := \bigcup_{l=0}^{k} \mathcal{D}_{k,l}$ and $\mathcal{A}_k := \bigcup_{l=0}^{k} \mathcal{A}_{k,l}$.

For $(w, D) \in \mathcal{D}_k$, where $D = \{d_1, \ldots, d_k\}_\leqslant$, let the *canonical factorization* of $w$ be

$$w = w_0 \cdots w_k$$

where, for each $i \in [k]$,

$$d_i = |w_0 \cdots w_{i-1}|.$$

For $\rho \in P$, define $o(\rho)$ to be the number $i \in [k]_0$ such that $\rho$ appears in $w_i$.

For $(v, A) \in \mathcal{A}_k$, where $A = \{a_1, \ldots, a_{r+k}\}_\leqslant$, let the *canonical factorization* of $v$ be

$$v = v_0 \cdots v_{r+k}$$

where, for each $j \in [r+k]$,

$$a_j = |v_0 \cdots v_{j-1}|.$$

For $\rho \in P$, define $q(\rho)$ to be the number $j \in [r+k]_0$ such that $\rho$ appears in $v_j$.

**Lemma 5.1.** *For $l \in \{n-r, \ldots, n-1\}$, $H_l = \emptyset$.*

**Proof.** There is a maximal chain

$$\rho_{i_0} < \cdots < \rho_{i_r}$$

where $i_0, \ldots, i_r \in [n]$. Then $i_0 < \cdots < i_r$ so any linear extension of $P$ contains at least $r$ ascents. $\square$

**Lemma 5.2.** *Let $k \in [n-1-r]_0$.*

(1) *Let* $(w, D) \in \mathcal{D}_k$ *and let* $w = w_0 \cdots w_k$ *be the canonical factorization. Then, for* $i \in [k]_0$, $w_i$ *is in increasing order.*
(2) *Let* $(v, A) \in \mathcal{A}_k$ *and let* $v = v_0 \cdots v_{r+k}$ *be the canonical factorization. Then, for* $j \in [r+k]_0$, $v_j$ *is in decreasing order.*

**Proof.** (1) This follows from the fact that $D(w) \subseteq D$. (2) This follows from the fact that $A(v) \subseteq A$. $\square$

**Lemma 5.3.** *Let* $k \in [n-1-r]_0$ *and suppose* $(v, A) \in \mathcal{A}_k$. *Then for* $\rho, \rho' \in P$ *such that* $\rho \leqslant \rho'$, *we have*

$$q(\rho) - r(\rho) \leqslant q(\rho') - r(\rho').$$

**Proof.** If $\rho < \rho'$, then $r(\rho) < r(\rho')$, so there is a saturated chain

$$\rho =: \rho_{i_{r(\rho)}} < \cdots < \rho_{i_{r(\rho')}} := \rho'$$

where $i_{r(\rho)}, \ldots, i_{r(\rho')} \in [n]$ with $i_{r(\rho)} < \cdots < i_{r(\rho')}$. By Lemma 5.2(2),

$$q(\rho) = q(\rho_{i_{r(\rho)}}) < \cdots < q(\rho_{i_{r(\rho')}}) = q(\rho')$$

and hence $q(\rho') - q(\rho) \geqslant r(\rho') - r(\rho)$. $\square$

**Corollary 5.4.** *Let* $k \in [n-1-r]_0$ *and suppose* $(v, A) \in \mathcal{A}_k$. *Then for all* $\rho \in P$, $0 \leqslant q(\rho) - r(\rho) \leqslant k$.

**Proof.** There exist $\rho', \rho'' \in P$ such that $\rho'' \leqslant \rho \leqslant \rho'$ and $r(\rho'') = 0$ and $r(\rho') = r$. By Lemma 5.3,

$$0 \leqslant q(\rho'') = q(\rho'') - r(\rho'') \leqslant q(\rho) - r(\rho) \leqslant q(\rho') - r(\rho') \leqslant (r+k) - r = k. \qquad \square$$

**Proposition 5.5.** *Fix* $k \in [n-1-r]_0$.

*Define a map* $\phi_k : \mathcal{D}_k \to \mathcal{A}_k$ *in the following manner. Given* $(w, D) \in \mathcal{D}_k$, *define a sequence of words* $v_0, \ldots, v_{r+k}$ *by letting* $\rho \in P$ *appear in the word* $v_{o(\rho)+r(\rho)}$ *and writing each word in decreasing order. Let* $v = v_0 \cdots v_{r+k}$, *and, for each* $j \in [r+k]$, *let*

$$a_j := |v_0 \cdots v_{j-1}|$$

*and let* $A = \{a_1, \ldots, a_{r+k}\}$. *Set* $\phi_k(w, D) = (v, A)$.

*Define a map* $\psi_k : \mathcal{A}_k \to \mathcal{D}_k$ *in the following manner. Given* $(v, A) \in \mathcal{A}_k$, *define a sequence of words* $w_0, \ldots, w_k$ *by letting* $\rho \in P$ *appear in the word* $w_{q(\rho)-r(\rho)}$ *and writing each word in increasing order. Let* $w = w_0 \cdots w_k$, *and, for each* $i \in [k]$, *let*

$$d_i := |w_0 \cdots w_{i-1}|$$

*and let* $D = \{d_1, \ldots, d_k\}$. *Set* $\psi_k(v, A) = (w, D)$.

*Then* $\phi_k$ *and* $\psi_k$ *are well-defined, mutually-inverse bijections.*

We illustrate this bijection in Section 6, which the reader might wish to read while going through the proof below.

**Proof.** In the first part of the proof, we show that $\phi_k$ is well defined. Select $(w, D) \in \mathcal{D}_k$. Let $w = w_0 \cdots w_k$ be the canonical factorization and let $v = v_0 \cdots v_{r+k}$ be as in the statement of the proposition. As $0 \leqslant o(\rho) + r(\rho) \leqslant k + r$ for each $\rho \in P$, $v$ contains each letter of $P$ exactly once.

We show that $v$ is a linear extension. Let $\rho, \rho' \in P$ be such that $\rho < \rho'$. Then $r(\rho) < r(\rho')$ and $o(\rho) \leqslant o(\rho')$ (since $w$ is a linear extension), so $o(\rho) + r(\rho) < o(\rho') + r(\rho')$. Thus $\rho$ appears to the left of $\rho'$ in $v$.

By Lemma 5.1, $|A(v)| \geqslant r$; and clearly $|A(v)| \leqslant r + k$ since $v_j$ is in decreasing order for each $j \in [r + k]_0$. Letting $l := |A(v)| - r$ we see that $v \in H_{n-1-r-l}$.

Because $D \subseteq [n - 1]$ and $n \geqslant 1$, we know $|w_0|, |w_k| \geqslant 1$. The first letter in $w_0$ must have rank 0 and so will be in $v_0$; the last letter in $w_k$ must have rank $r$ and so will be in $v_{r+k}$. Hence $A \subseteq [n - 1]$. Because each of $v_0, \ldots, v_{r+k}$ is in decreasing order, $A(v) \subseteq A$. Hence $(v, A) \in \mathcal{A}_{k,l}$. Note that $v_0 \cdots v_{r+k}$ is the canonical factorization of $v$.

In the second part of the proof, we show that $\psi_k$ is well defined. Select $(v, A) \in \mathcal{A}_k$. Let $v = v_0 \cdots v_{r+k}$ be the canonical factorization and let $w = w_0 \cdots w_k$ be as in the statement of the proposition. These words are well defined by Corollary 5.4; $w$ contains each letter of $P$ exactly once.

We show that $w$ is a linear extension. Let $\rho, \rho' \in P$ be such that $\rho < \rho'$. By Lemma 5.3 and the fact that $w_0, \ldots, w_k$ are in increasing order, $\rho$ appears to the left of $\rho'$ in $w$.

The fact that $w_0, \ldots, w_k$ are in increasing order also says that $D(w) \subseteq D$. Because $A \subseteq [n - 1]$ and $n \geqslant 1$, we know $|v_0|, |v_{r+k}| \geqslant 1$. The first letter of $v_0$ must have rank 0 and so will be in $w_0$; the last letter of $v_{r+k}$ must have rank $r$ and so will be in $w_k$. Hence $D \subseteq [n-1]$ and thus $(w, D) \in \mathcal{D}_k$. Note that $w_0 \cdots w_k$ is the canonical factorization of $w$.

Now again select $(w, D) \in \mathcal{D}_k$ and let $(v, A) = \phi_k(w, D)$ and $(w', D') = \psi_k(v, A)$. Let $w = w_0 \cdots w_k$, $w' = w'_0 \cdots w'_k$, and $v = v_0 \cdots v_{r+k}$ be the canonical factorizations of $w$, $w'$, and $v$, respectively. For $i \in [k]_0$, $\rho \in P$ appears in $w_i$ if and only if it appears in $v_{i+r(\rho)}$ if and only if it appears in $w'_i$; thus $w_i = w'_i$. Hence $w = w'$ and $D = D'$.

Select $(v, A) \in \mathcal{A}_k$ and let $(w, D) = \psi_k(v, A)$ and $(v', A') = \phi_k(w, D)$. Let $v = v_0 \cdots v_{r+k}$, $v' = v'_0 \cdots v'_{r+k}$, and $w = w_0 \cdots w_k$ be the canonical factorizations of $v$, $v'$, and $w$, respectively. For $j \in [r + k]_0$, $\rho \in P$ appears in $v_j$ if and only if it appears in $w_{j-r(\rho)}$ if and only if it appears in $v'_j$; thus $v_j = v'_j$. Hence $v = v'$ and $A = A'$. □

**Lemma 5.6.** *Let $k, l \in [n - 1 - r]_0$ where $l \leqslant k$. Suppose there exists a bijection $\Phi_{l,l} : \mathcal{D}_{l,l} \to \mathcal{A}_{l,l}$ with inverse $\Psi_{l,l} : \mathcal{A}_{l,l} \to \mathcal{D}_{l,l}$.*

*Define a map*

$$\Phi_{k,l} : \mathcal{D}_{k,l} \to \mathcal{A}_{k,l}$$

*as follows: for all $(w, D) \in \mathcal{D}_{k,l}$, $\Phi_{k,l}(w, D) := (v, A)$ where*

$$\bigl(v, A(v)\bigr) = \Phi_{l,l}\bigl(w, D(w)\bigr) \quad \text{and} \quad A = A(v) \cup \bigl[D \setminus D(w)\bigr]$$

(*a union of multisets*).

*Define a map*

$$\Psi_{k,l} : \mathcal{A}_{k,l} \to \mathcal{D}_{k,l}$$

*as follows*: *for all* $(v, A) \in \mathcal{A}_{k,l}$, $\Psi_{k,l}(v, A) = (w, D)$ *where*

$$\big(w, D(w)\big) = \Psi_{l,l}\big(v, A(v)\big) \quad \textit{and} \quad D = D(w) \cup \big[A \setminus A(v)\big]$$

(*a union of multisets*).

*Then* $\Phi_{k,l}$ *and* $\Psi_{k,l}$ *are well-defined, mutually-inverse bijections.*

**Proof.** First we show that $\Phi_{k,l}$ is well defined. With $(w, D) \in \mathcal{D}_{k,l}$ as above, $|A| = |A(v)| + |D| - |D(w)| = r + l + k - l = r + k$.

Next, we show that $\Psi_{k,l}$ is well defined. With $(v, A) \in \mathcal{A}_{k,l}$ as above, $|D| = |D(w)| + |A| - |A(v)| = l + r + k - (r + l) = k$.

Now suppose

$$(w, D) \in \mathcal{D}_{k,l}, \quad (v, A) = \Phi_{k,l}(w, D), \quad \text{and} \quad \big(w', D'\big) = \Psi_{k,l}(v, A).$$

Clearly $w = w'$ (because $\Phi_{l,l}$ and $\Psi_{l,l}$ are inverses). Also,

$$\begin{aligned} D' &= D(w) \cup \big[\big(A(v) \cup \big[D \setminus D(w)\big]\big) \setminus A(v)\big] \\ &= D(w) \cup \big(D \setminus D(w)\big) = D \end{aligned}$$

since $D(w) \subseteq D$.

Finally, suppose

$$(v, A) \in \mathcal{A}_{k,l}, \quad (w, D) = \Psi_{k,l}(v, A), \quad \text{and} \quad \big(v', A'\big) = \Phi_{k,l}(w, D).$$

Clearly $v = v'$. Also,

$$\begin{aligned} A' &= A(v) \cup \big[\big(D(w) \cup \big[A \setminus A(v)\big]\big) \setminus D(w)\big] \\ &= A(v) \cup \big(A \setminus A(v)\big) = A \end{aligned}$$

since $A(v) \subseteq A$. $\square$

**Lemma 5.7** (Involution Principle, q.v. [4,20, §2.6]). *Let* $X$, $Y$, $X'$, *and* $Y'$ *be finite sets with* $X \cap Y = \emptyset = X' \cap Y'$. *Let* $\Phi_X : X \to X'$ *and* $\phi : X \cup Y \to X' \cup Y'$ *be bijections with inverses* $\Psi_X : X' \to X$ *and* $\psi : X' \cup Y' \to X \cup Y$, *respectively.*

*Define a map* $\Phi_Y : Y \to Y'$ *as follows. For all* $y \in Y$, *let* $t \geqslant 0$ *be the smallest non-negative integer such that*

$$\big((\phi \circ \Psi_X)^t \circ \phi\big)(y) =: y' \in Y'$$

(*such a* $t$ *must exist*) *and let* $\Phi_Y(y) := y'$.

*Define a map $\Psi_Y : Y' \to Y$ as follows. For all $y' \in Y'$, let $t \geqslant 0$ be the smallest non-negative integer such that*

$$\big((\psi \circ \Phi_X)^t \circ \psi\big)\big(y'\big) =: y \in Y$$

*and let $\Psi_Y(y') := y$.*

*Then $\Phi_Y$ and $\Psi_Y$ are well-defined, mutually-inverse bijections.*

**Theorem 5.8.** *Let $P$ be a finite ranked poset of cardinality $n \geqslant 2$ and rank $r$. Let $k \in [n-1-r]_0$.*

*Construct an explicit bijection $\Phi_{k,k} : H_k \to H_{n-1-r-k}$ in the following manner.* (*We identify $H_l$ with $\mathcal{D}_{l,l}$ and $H_{n-1-r-l}$ with $\mathcal{A}_{l,l}$ for all $l \leqslant k$.*)

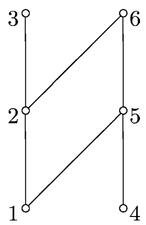
*If $k = 0$, use the map $\phi_0$ of Proposition* 5.5.

*If $k \geqslant 1$, first construct the bijections $\Phi_{l,l} : H_l \to H_{n-1-r-l}$ for $l \in [k-1]_0$; then construct the bijections $\Phi_{k,l} : \mathcal{D}_{k,l} \to \mathcal{A}_{k,l}$ as per Lemma* 5.6. *Use the involution principle of Lemma* 5.7 *with $X = \bigcup_{l=0}^{k-1} \mathcal{D}_{k,l}$, $Y = H_k$, $X' = \bigcup_{l=0}^{k-1} \mathcal{A}_{k,l}$, $Y' = H_{n-1-r-k}$, $\Phi_X = \bigcup_{l=0}^{k-1} \Phi_{k,l}$, and $\phi = \phi_k$* (*the map of Proposition* 5.5).

Thus we solve the problem of Stanley from the 1981 Banff Conference on Ordered Sets.

## 6. An example of the bijection solving Stanley's problem

Consider the ranked poset of Fig. 6.1 with $n = 6$ and $r = 2$. Its $h$-vector is $(1, 6, 6, 1)$; see Table 6.1 for all of its linear extensions.




$h_0 = 1$ $\quad h_1 = 6$ $\quad h_2 = 6$ $\quad h_3 = 1$

$n = 6$ $\quad r = 2$

Fig. 6.1. A poset used to illustrate Theorem 5.8.

Table 6.1
Linear extensions of the poset of Fig. 6.1

| $H_0$ | $H_1$ | $H_2$ | $H_3$ |
|---|---|---|---|
| 123456 | 124356 | 415236 | 415263 |
| | 124536 | 412563 | |
| | 124563 | 412536 | |
| | 142356 | 145263 | |
| | 145236 | 142563 | |
| | 412356 | 142536 | |

For $k = 0$ we have

$$\mathcal{D}_{0,0} = \{(123456, \emptyset)\} \quad \text{and} \quad \mathcal{A}_{0,0} = \{(41\ 52\ 63, 24)\}.$$

(For clarity, we leave out the braces and commas when listing the multisets.) We leave it to the reader to guess the map $\phi_0$ of Proposition 5.5 (and hence the map $\Phi_{0,0}$ of Theorem 5.8).

For $k = 1$, we have

$$\mathcal{D}_{1,0} = \{(1\ 23456, 1), (12\ 3456, 2), (123\ 456, 3), (1234\ 56, 4), (12345\ 6, 5)\}$$

and

$$\begin{aligned}\mathcal{D}_{1,1} = \{&(124\ 356, 3), (1245\ 36, 4), (12456\ 3, 5),\\ &(14\ 2356, 2), (145\ 236, 3), (4\ 12356, 1)\}.\end{aligned}$$

We also have

$$\begin{aligned}\mathcal{A}_{1,0} = \{&(4\ 1\ 52\ 63, 124), (41\ \ 52\ 63, 224), (41\ 5\ 2\ 63, 234),\\ &(41\ 52\ \ 63, 244), (41\ 52\ 6\ 3, 245)\}\end{aligned}$$

and

$$\begin{aligned}\mathcal{A}_{1,1} = \{&(41\ 52\ 3\ 6, 245), (41\ 2\ 5\ 63, 234), (41\ 2\ 53\ 6, 235),\\ &(1\ 4\ 52\ 63, 124), (1\ 42\ 5\ 63, 134), (1\ 42\ 53\ 6, 135)\}.\end{aligned}$$

We describe the map $\phi_1$ of Proposition 5.5 by using spaces to delineate the factors in the canonical factorizations. (See Table 6.2.)

The map $\Phi_{1,0}$ of Lemma 5.6 is given by

Table 6.2
The map $\phi_1$

| $D$ | $w_0$ | $w_1$ | $\xrightarrow{\phi_1}$ | $v_0$ | $v_1$ | $v_2$ | $v_3$ | $A$ |
|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 23456 | | 1 | 4 | 52 | 63 | 124 |
| 2 | 12 | 3456 | | 1 | 42 | 5 | 63 | 134 |
| 3 | 123 | 456 | | 1 | 42 | 53 | 6 | 135 |
| 4 | 1234 | 56 | | 41 | 2 | 53 | 6 | 235 |
| 5 | 12345 | 6 | | 41 | 52 | 3 | 6 | 245 |
| 3 | 124 | 356 | | 41 | 2 | 5 | 63 | 234 |
| 4 | 1245 | 36 | | 41 | 52 | | 63 | 244 |
| 5 | 12456 | 3 | | 41 | 52 | 6 | 3 | 245 |
| 2 | 14 | 2356 | | 41 | | 52 | 63 | 224 |
| 3 | 145 | 236 | | 41 | 5 | 2 | 63 | 234 |
| 1 | 4 | 12356 | | 4 | 1 | 52 | 63 | 124 |

$$(1\ 23456, 1) \xrightarrow{\Phi_{1,0}} (4\ 1\ 52\ 63, 124)$$

$$(12\ 3456, 2) \xrightarrow{\Phi_{1,0}} (41\ \ 52\ 63, 224)$$

$$(123\ 456, 3) \xrightarrow{\Phi_{1,0}} (41\ 5\ 2\ 63, 234)$$

$$(1234\ 56, 4) \xrightarrow{\Phi_{1,0}} (41\ 52\ \ 63, 244)$$

$$(12345\ 6, 5) \xrightarrow{\Phi_{1,0}} (41\ 52\ 6\ 3, 245)$$

Finally, we can compute $\Phi_{1,1}$ using the involution principle:

$$\begin{array}{ccccccc}
124356 & \xrightarrow{\phi_1} & 412563 & & & & \\
124536 & \xrightarrow{\phi_1} & (41\ 52\ \ 63, 244) & \xrightarrow{\Psi_{1,0}} & (1234\ 56, 4) & \xrightarrow{\phi_1} & 412536 \\
124563 & \xrightarrow{\phi_1} & (41\ 52\ 6\ 3, 245) & \xrightarrow{\Psi_{1,0}} & (12345\ 6, 5) & \xrightarrow{\phi_1} & 415236 \\
142356 & \xrightarrow{\phi_1} & (41\ \ 52\ 63, 224) & \xrightarrow{\Psi_{1,0}} & (12\ 3456, 2) & \xrightarrow{\phi_1} & 142563 \\
145236 & \xrightarrow{\phi_1} & (41\ 5\ 2\ 63, 234) & \xrightarrow{\Psi_{1,0}} & (123\ 456, 3) & \xrightarrow{\phi_1} & 142536 \\
412356 & \xrightarrow{\phi_1} & (4\ 1\ 52\ 63, 124) & \xrightarrow{\Psi_{1,0}} & (1\ 23456, 1) & \xrightarrow{\phi_1} & 145263
\end{array}$$

Hence the bijection

$$\Phi_{1,1} : H_1 \to H_2$$

is given by

$$124356 \xrightarrow{\Phi_{1,1}} 412563$$

$$124536 \xrightarrow{\Phi_{1,1}} 412536$$

$$124563 \xrightarrow{\Phi_{1,1}} 415236$$

$$142356 \xrightarrow{\Phi_{1,1}} 142563$$

$$145236 \xrightarrow{\Phi_{1,1}} 142536$$

$$412356 \xrightarrow{\Phi_{1,1}} 145263$$

## 7. The future of an injection

While we have solved the problem of Stanley, our results could be improved in three ways. First, our bijection works for an arbitrary ranked poset with an arbitrary natural labelling, but there may be a more "natural" bijection for particular types of ranked posets with particular natural labellings. So it would still be satisfying to construct the bijection for a product of two chains. Second, the part of our bijection where we invoke the involution principle can probably be described even more explicitly in a manner reminiscent of jeu de taquin (although, needless to say, without the same far-reaching consequences).

Third, one could perhaps prove that $h_k \leqslant h_{n-1-r-k}$ for an arbitrary (not necessarily ranked) poset of cardinality $n$ and height $r$ ($k \leqslant \lfloor \frac{n-1-r}{2} \rfloor$) by refining our solution to Stanley's problem.

## A. Poset menagerie



$h_0 = 1 \quad h_1 = 1$

$n = 2 \quad r = 0$

(a)



$h_0 = 1$

$n = 2 \quad r = 1$

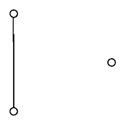
(b)



$h_0 = 1 \quad h_1 = 4 \quad h_2 = 1$

$n = 3 \quad r = 0$

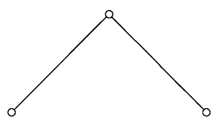
(c)



$h_0 = 1 \quad h_1 = 2$

$n = 3 \quad r = 1$

(d)



$h_0 = 1 \quad h_1 = 1$

$n = 3 \quad r = 1$

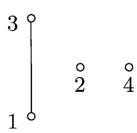
(e)



$h_0 = 1 \quad h_1 = 11 \quad h_2 = 11 \quad h_3 = 1$

$n = 4 \quad r = 0$

(f)

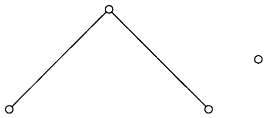



$h_0 = 1 \quad h_1 = 7 \quad h_2 = 4$

$n = 4 \quad r = 1$

(g)

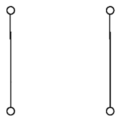
Fig. A.1. Examples of $h$-vectors.



$h_0 = 1$ $h_1 = 5$ $h_2 = 2$

$n = 4$ $r = 1$

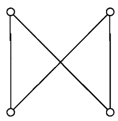
(h)



$h_0 = 1$ $h_1 = 4$ $h_2 = 1$

$n = 4$ $r = 1$

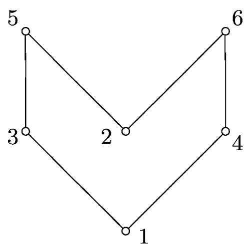
(i)



$h_0 = 1$ $h_1 = 2$ $h_2 = 1$

$n = 4$ $r = 1$

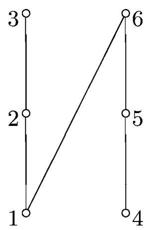
(j)




$h_0 = 1$ $h_1 = 8$ $h_2 = 11$ $h_3 = 2$

$n = 6$ $r = 2$

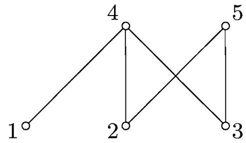
(k)




$h_0 = 1$ $h_1 = 8$ $h_2 = 9$ $h_3 = 1$

$n = 6$ $r = 2$

(l)

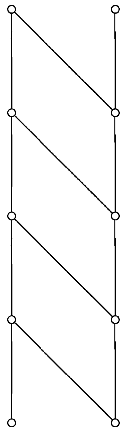



$h_0 = 1$ $h_1 = 6$ $h_2 = 6$ $h_3 = 1$

$n = 5$ $r = 1$

(m)

Fig. A.1. (Continued.)



$h_0 = 1 \quad h_1 = 15 \quad h_2 = 50$

$h_3 = 50 \quad h_4 = 15 \quad h_5 = 1$

$n = 10 \quad r = 4$

(n)

Fig. A.1. (Continued.)

Table A.1
Linear extensions of $\mathbf{1}+\mathbf{1}+\mathbf{1}+\mathbf{1}$ and $\mathbf{1}+\mathbf{1}+\mathbf{2}$ (the latter unmarked)

| $H_0$ | $H_1$ | $H_2$ | $H_3$ |
|---|---|---|---|
| 1234 | 2134 | 4312∗ | 4321∗ |
| | 3124∗ | 4213 | |
| | 4123 | 3214∗ | |
| | 1324 | 4231∗ | |
| | 1423 | 3241∗ | |
| | 2314∗ | 4132 | |
| | 2413 | 3142∗ | |
| | 3412∗ | 2143 | |
| | 1243 | 3421∗ | |
| | 1342 | 2431∗ | |
| | 2341∗ | 1432 | |

Table A.2
Linear extensions of the poset of Fig. A.1(k)

| $H_0$ | $H_1$ | $H_2$ | $H_3$ |
|---|---|---|---|
| 123456 | 123465 | 124365 | 143265 |
| | 123546 | 132465 | 214365 |
| | 124356 | 132546 | |
| | 124635 | 134265 | |
| | 132456 | 142365 | |
| | 134256 | 143256 | |
| | 142356 | 213465 | |
| | 213456 | 142635 | |
| | | 213546 | |
| | | 214356 | |
| | | 214635 | |

Table A.3
Linear extensions of the posets of Figs. 2.1 and A.1(l) (the latter unmarked)

| $H_0$ | $H_1$ | $H_2$ | $H_3$ |
|---|---|---|---|
| 123456 | 124356 | 451623 | 415263 |
| | 124536 | 415623 | |
| | 124563 | 415236 | |
| | 142356 | 451263 | |
| | 145236 | 412563 | |
| | 145623 | 412536 | |
| | 412356 | 145263 | |
| | 451236 | 142563 | |
| | 456123∗ | 142536 | |

Table A.4
Linear extensions of the poset of Fig. A.1(m)

| $H_0$ | $H_1$ | $H_2$ | $H_3$ |
|---|---|---|---|
| 12345 | 13245 | 13254 | 32154 |
| | 21345 | 21354 | |
| | 12354 | 32145 | |
| | 31245 | 31254 | |
| | 23145 | 23154 | |
| | 23514 | 32514 | |

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