Chapter 14
How Al Qaeda Can Use Order Theory to Evade or Defeat U.S. Forces
The Case of Binary Posets

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Abstract Terrorist cells are modeled as finite partially ordered sets. This paper determines the structure of the terrorist cell most likely to remain intact if a subset of its members is captured at random, provided that the cell has a single leader and no member has more than 2 immediate subordinates.

14.1 Introduction

I arrived at Ted K.'s cell. "Cell" was the wrong word: As the door swung open, I walked into what you would think of as a plush apartment. All that would strike you as strange was the absence of any windows.

"I still can't get over how well they take care of you here," I said, shaking my head. "Is that flat-screen TV new? I'm surprised they let you watch the news."

Ted was sitting at a large glass table, papers with his neat handwriting littering the surface—ordered chaos. His grey-black beard and hair were as wild as ever. "Don't worry: it's just for playing video games."

I sat down, folded my hands and said, "You have something for me?"

Ted smiled, the wrinkles around his eyes deepening. "As you know, for the last five years I have been working on the problem of creating the perfect terrorist cell."

"From inside prison?" I asked with a cold smile.

"To be precise, I want to determine the structure of the terrorist cell that can best withstand your government's attempts to destroy it—a cell that, if a certain number of its members were to be captured at random, would still have the best possible chance of being able to carry out its mission."

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“And why would you want to do that?”

“To settle a score. That fellow from the NSA said my research wasn’t useful. Wait till Al Qaeda gets a hold of it: Then your military will wish it had given me funding when they had the chance.”

I usually let Ted go on a while with his *the-fools-I’ll-show-them-all* monologue, but today I had little time. “And you’ve succeeded?”

Ted’s shoulders sagged slightly. “Not quite, but I’m getting close.

“You remember that we model terrorist cells as partially ordered sets, or *posets*, which are essentially organization charts where commands flow from the top to the bottom.” He sketched a diagram, which he labeled, “Figure 14.1.”

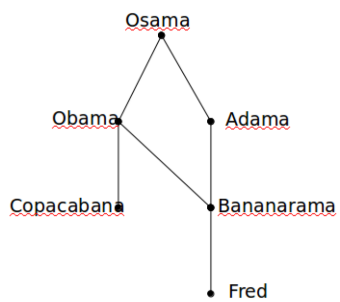

Fig. 14.1

“In this example, Osama sits at the top of the organization. He can pass plans down to his immediate subordinates, Obama”—I rolled my eyes—“and Adama, but only them. Osama must rely on Obama and Adama to pass his orders further down the organization: Adama to Bananarama, his only immediate subordinate, and Obama to either Bananarama or Copacabana.

“Eventually Osama’s orders reach the foot soldiers, Copacabana and Fred, who are the people who will actually carry out the attacks.

“Your government’s goal,” Ted continued, “is to arrest—”

“Or kill.”

“Arrest or kill enough terrorists so Osama’s plans cannot reach the foot soldiers. For example, you could capture Osama himself, or both Obama and Adama, or both Obama and Fred.”

“But not Copacabana and Adama,” I said impatiently, “because then Osama’s orders could still get to a foot soldier, Fred, via Obama and Bananarama. I read Fraly’s monograph, you know.”

Ted smiled wanly. "Then you know that the subgroups of terrorists you want to capture are called *cutsets*. So the cell with the fewest cutsets will be the hardest for your government to disrupt."

Ted sat back and put his hands behind his head. "Of course, the cell with a single leader that has the fewest cutsets is clearly a spider: one leader with everyone else as his immediate subordinate." He called this "Figure 14.2." "To prevent the leader from passing his plans to at least one foot soldier, you either have to capture him or capture every foot soldier.

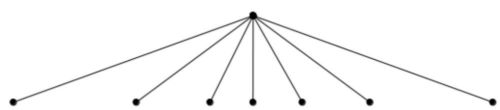

Fig. 14.2: A spider.

"But this is unrealistic," Ted continued, "because can one man really supervise 8 or 9 or 18 different people? It is much more practical," Ted said, "to suppose there is a bound, *b*—say 3 or 4—on the number of immediate subordinates a terrorist can have.

"If the cell is a tree—that is, if the cell has one leader and no one has more than one immediate superior, like Figure 14.3—then some computer scientists in Montreal found that the best tree structure looks like this." He drew Figure 14.4. "You have a leader with exactly *b* immediate subordinates, all of them foot soldiers except for one, who has *b* immediate subordinates, and so on, until you run out of men." He sighed.

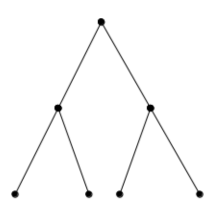

Fig. 14.3: A binary tree with 7 terrorists.

"The problem is, there are lots of possible terrorist cell structures that are *not* trees."

"Figure 14.1, for instance," I chimed in.

"Still," he said, "Fraly looked at all binary cells—"

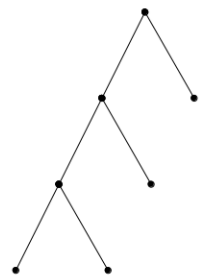

Fig. 14.4: The perfect binary terrorist tree with 7 members.

"'Binary'?" I asked.

"Where b equals 2," Ted clarified. "Fraly showed in his monograph that the best binary cell with a single leader and at most 6 terrorists was always one of the special trees he had discovered, the ones the Montreal group later considered, and Fraly posed a problem." He paused. "But now I have settled the issue."

I stared at him, narrowing my eyes.

"In the binary case," Ted said quietly.

14.2 Proof That the Optimal Binary Terrorist Cell with One Leader Is a Pure Fishbone Poset

For order theory terminology, refer to Davey and Priestley 2002 and Farley 2003.

Definition 1. Let P be a finite poset. For p in P, let LC(p) be the set of lower covers of p.

Definition 2. Let P be a finite poset. Let k be a natural number. A k-*cutset* is a k-element subset that intersects every maximal chain of P. Let Cut(P,k) be the set of k-cutsets of P and let cut(P,k) equal $|\mathrm{Cut}(P,k)|$. A subset C of P is a *cutset* if it is a $|C|$-cutset.

Observation 3. Let P be a finite poset with greatest element $\top$. Assume LC($\top$)= $\{t,x\}$. Then $\{t,x\}$ is a cutset, and, if LC(t) and LC(x) are non-empty, $\{t\}\cup\mathrm{LC}(x)$, $\{x\}\cup\mathrm{LC}(t)$, and $\mathrm{LC}(t)\cup\mathrm{LC}(x)$ are cutsets. $\square$

Definition 4. Let P be a finite poset. Let b be a natural number. We say P is b-*ary* if, for all p in P, $|\mathrm{LC}(p)| \leq b$; if $b = 2$, we call P *binary*.

Definition 5. For a natural number n, define the poset FP(n) as follows:

$$\begin{aligned} \mathrm{FP}(0) &:= \emptyset \\ \mathrm{FP}(1) &:= \mathbf{1}, \text{ the one-element poset} \\ \mathrm{FP}(k+2) &:= [\mathrm{FP}(k)+\mathbf{1}] \oplus \mathbf{1} \text{ for all } k \geq 0. \end{aligned}$$

Example 6. The posets FP(n) are shown in Figure 14.5 for $1 \leq n \leq 6$.

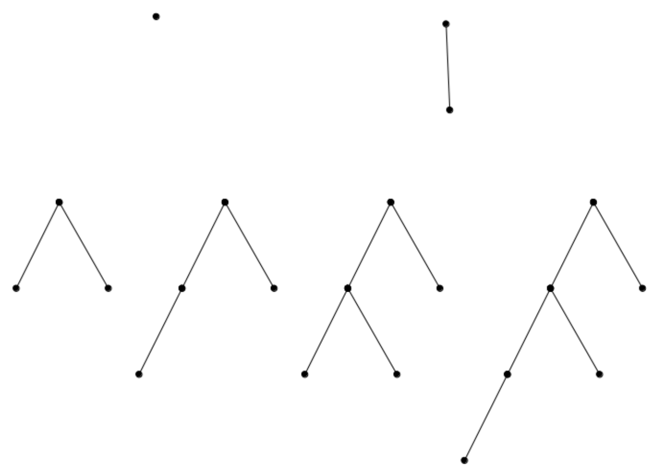

Fig. 14.5: The posets $FP(n)$ for $1 \leq n \leq 6$.

Note 7. For $n \geq 1$, FP(n) is called in Farley 2007, Definition A0.1 the *pure fishbone poset of type* $(\frac{n+1}{2}, \frac{n-1}{2}; 1, 2, \ldots, \frac{n-1}{2}; n)$ if n is odd and $(\frac{n+2}{2}, \frac{n-2}{2}; 1, 2, \ldots, \frac{n-2}{2}; n)$ if n is even.

Proposition 8. For $n \geq 3$, cut(FP(n),1)=1.

For $n \geq 3$, cut(FP(n),2)= $\begin{cases} n+1 & \text{if } n = 4 \\ n & \text{if } n \neq 4. \end{cases}$

For $n \geq 5$, cut(FP(n),3)= $\begin{cases} \binom{n-1}{2} + \binom{n-3}{1} + 2 & \text{if } n = 6 \\ \binom{n-1}{2} + \binom{n-3}{1} + 1 & \text{if } n \neq 6. \end{cases}$

Proof. This follows from Corollaries A0.1 and A0.2 of Farley 2007 or by direct analysis. $\square$

Corollary 9. Let P be a finite poset with a greatest element $\top$ that has exactly two lower covers, t and x.

(1) Let $a = 1$ if $|P| = 4$ and let $a = 0$ otherwise. If

$$|\{C \in \mathrm{Cut}(P,2) : \top \notin C \text{ and } C \neq \{t,x\}| > a$$

then cut(P,2) $>$ cut(FP($|P|$),2).

(2) Let $b=2$ if $|P|=6$ and let $b=1$ otherwise. If $|\mathrm{LC}(t) \cup \mathrm{LC}(x)| \geq 2$ and

$$|\{C \in \mathrm{Cut}(P,3) : \top \notin C \text{ and } \{t,x\} \not\subseteq C\}| > b$$

Then cut(P,3) $>$ cut(FP($|P|$),3).

Proof. (1) The cardinality of $\{C \in \mathrm{Cut}(P,2) : \top \in C\}$ is $n-1$, so by Observation 3,

$$\mathrm{cut}(P,2) > n-1+1+a.$$

Use Proposition 8.

(2) The cardinality of $\{C \in \mathrm{Cut}(P,3) : \top \in C\}$ is $\binom{n-1}{2}$;

$$|\{C \in \mathrm{Cut}(P,3) : \top \notin C \text{ and } \{t,x\} \subseteq C\}| = \binom{n-3}{1}$$

by Observation 3; and cut(P,3)$>$ $\binom{n-1}{2}+\binom{n-3}{1}+b$, so by Proposition 8 we are done. □

Lemma 10. Let P be a finite binary poset with greatest element $\top$. Assume that $|P|$ is at least 3 and that for k equal to 1, 2, or 3, cut(P,k)$\leq$cut(FP($|P|$),k).

Then there exists a lower cover x of $\top$ such that $P \setminus \{x,\top\}$ has a greatest element and

$$P = [P \setminus \{x,\top\} + \{x\}] \oplus \{\top\}.$$

Proof. Since $|P|$ is at least 3, $|\mathrm{LC}(\top)|$ is at least 1. If $|\mathrm{LC}(\top)|$=1, then by Observation 3, cut(P,1)$\geq$ 2, contradicting Proposition 8.

Since P is binary, let LC($\top$)=$\{t,x\}$ where $t \neq x$. Assume for a contradiction that LC(t) and LC(x) are non-empty.

If $|\mathrm{LC}(t)| = 1 = |\mathrm{LC}(x)|$, then Observation 3 and Corollary 9(1) contradict Proposition 8.

Without loss of generality, $|\mathrm{LC}(t)| = 2$. If $|\mathrm{LC}(x)| = 1$, then, since $|P| \neq 4$, Observation 3 contradicts Corollary 9(1).

If $|\mathrm{LC}(t) \cap \mathrm{LC}(x)|$ equals 2 or 1, then Observation 3 contradicts Corollary 9(1) or (2). Thus LC(t)$\cap$LC(x)$=\emptyset$, so $|P| \neq 6$ and hence Observation 3 contradicts Corollary 9(2).

Without loss of generality, LC(x)$=\emptyset$. Then t is the greatest element of $P \setminus \{x,\top\}$ and $x \parallel y$ for all y in $P \setminus \{x,\top\}$. □

Lemma 11. Let Q be a non-empty finite poset. Let x and $\top$ be distinct elements not in Q and let $P = (Q+\{x\}) \oplus \{\top\}$. Then for all $k \geq 0$,

$$\mathrm{Cut}(P,k+1) = \{B \cup \{x\} : B \in \mathrm{Cut}(Q,k)\} \cup \{D \cup \{x,\top\} \colon D \subseteq Q \text{ and } |D| = k-1\}$$

$$\cup \{E \cup \{\top\} \colon E \subseteq Q \text{ and } |E| = k\}.$$

Proof. Let $C \in \mathrm{Cut}(P, k+1)$. If x and $\top$ belong to C, then let D equal $C \setminus \{x, \top\}$. If $\top$ belongs to C but not x, then let E equal $C \setminus \{\top\}$. If $\top$ does not belong to C, then x belongs to C since C is a cutset. Let B equal $C \setminus \{x\}$.

If B is not a k-cutset of Q, then there is a maximal chain N of Q such that $N \cap B = \emptyset$. Then $M := N \cup \{\top\}$ is a maximal chain of P since N is non-empty, so $M \cap C \neq \emptyset$, i.e., $\emptyset \neq N \cap C = N \cap B$, a contradiction.

Conversely, if F is a subset of Q, then $F \cup \{\top\}$ and $F \cup \{x, \top\}$ are cutsets of P. If B is a k-cutset of Q, let C equal $B \cup \{x\}$. If $C \notin \mathrm{Cut}(P, k+1)$, then there exists a maximal chain M of P such that $M \cap C = \emptyset$. Clearly $x \notin M$ and $\top \in M$, and $M \setminus \{\top\}$ is a maximal chain of Q, so $M \cap C \supseteq M \cap B \neq \emptyset$, a contradiction. □

Theorem 12. Let P be a finite binary poset with a greatest element such that, for all $k \geq 0$, cut(P,k)$\leq$cut(FP($|P|$),k).

Then P is isomorphic to FP($|P|$).

Proof (by induction on $|P|$). We may assume that P has at least 3 elements. By Lemma 10, there exist x in P and a subset Q of P with a greatest element such that $P = (Q + \{x\}) \oplus \{\top\}$, where $\top$ is the greatest element of P. Clearly Q is binary.

By Lemma 11, for all $k \geq 0$, cut(Q,k)$\leq$cut(FP($|P|-2$),k), so by induction

$$Q \cong \mathrm{FP}(|P|-2).$$

Hence P is isomorphic to FP($|P|$). □

14.3 Conclusion

"So what's that mean in English?" I pleaded.

Ted sighed. "It means that if a terrorist cell has a single leader, and each terrorist has at most 2 immediate subordinates, then the cell structure most likely to succeed if some of the terrorists are captured at random looks like something in Figures 14.4 and 14.5."

I shuffled my papers and stood up from the glass table. "Okay, we're done here," I said: "You're never getting out." I smiled.

Ted was unperturbed. "Wait, I'm not dangerous yet. I haven't figured out how terrorists should organize their cells if each man can have 3 or more subordinates. There's still work to be done. You still have time to support my research before it's too late for your government, before Al Qaeda gets a hold of my work and uses it to evade or defeat your government's forces."

"I'm not worried," I lied. I walked to the door of his cell, pausing with a shock as I opened it.

Man! Even the door handles were nice.

References

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[5] Jonathan David Farley, "Two Theoretical Research Questions Concerning the Structure of the Perfect Terrorist Cell," in Nasrullah Memon, Jonathan David Farley, David L. Hicks, and Torben Rosenørn (eds.), *Mathematical Methods in Counterterrorism* (Springer Verlag, Vienna, 2009), 91–103.