Algebra univers. 45 (2001) 375 – 381
0002–5240/01/040375 – 07 \$ 1.50 + 0.20/0

Algebra Universalis

Coproducts of bounded distributive lattices: cancellation

A Problem from the 1981 Banff Conference on Ordered Sets

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Abstract. Let $L * M$ denote the coproduct of the bounded distributive lattices L and M. At the 1981 Banff Conference on Ordered Sets, the following question was posed: What is the largest class $\mathcal{L}$ of finite distributive lattices such that, for every non-trivial Boolean lattice B and every $L \in \mathcal{L}$, $B * L = B * L'$ implies $L = L'$?

In this note, the problem is solved.

1. Introduction

A *Post algebra* is a bounded distributive lattice (with least element 0 and greatest element 1) with a non-trivial finite chain $0 = c_0 < \cdots < c_{n-1} = 1$ and a Boolean sublattice B such that every element has a unique representation $\bigvee_{i=1}^{n-1}(b_i \wedge c_i)$ where $b_1 \geqslant \cdots \geqslant b_{n-1}$ are in B. The chain is called the *chain of constants*. It is well known that Post algebras are exactly the coproducts $B * C$ of non-trivial Boolean lattices B and non-trivial finite chains C in the category of bounded distributive lattices (see [13], Theorem 2). It is also well known that the chain of constants in a Post algebra P is unique in the following sense: If B is a Boolean sublattice of P and C and C' are finite subchains containing 0 and 1 such that $P = B * C = B * C'$, then $C = C'$. That is, C and C' are equal as subsets, not simply isomorphic ([7], Theorem 2.1). More precisely, if $\iota_B : B \hookrightarrow B * C, \iota_C : C \hookrightarrow B * C, \iota'_B : B \hookrightarrow B * C'$, and $\iota_{C'} : C' \hookrightarrow B * C'$ are the natural monomorphisms and $\Psi : B * C \cong B * C'$ an isomorphism, then the image of $\Psi \circ \iota_C$ is the image of $\iota_{C'}$.

Balbes and Dwinger proved that if B and B' are non-trivial Boolean lattices and C and C' non-trivial bounded chains, then $B * C = B' * C'$ implies $B = B'$ and $C \cong C'$ ([1],

Presented by Professor Willem J. Blok.

Received March 2, 1999; accepted in final form July 10, 2000.

2000 *Mathematics Subject Classification*: 06D25, 54F05, 06E15, 54A05, 54B10, 54D05.

Key words and phrases: (generalized) Post algebra, distributive lattice, Boolean lattice, coproduct, Priestley duality, (partially) ordered topological space.

The author would like to thank Dr. H. A. Priestley for comments regarding the presentation of this paper. He would also like to thank Dr. Stephen Comer for bringing to his attention reference [2], and the referee for his or her comments. The author acknowledges support from the Mathematical Sciences Research Institute in Berkeley, California and National Science Foundation grant 9971352.

Theorem 1). Moreover, for a given non-trivial bounded chain C, $B * C = B * C'$ implies $C = C'$ for all non- trivial Boolean lattices B and all bounded chains C' if and only if C is *rigid*, that is, has just one automorphism ([1], Theorem 6). Comer and Dwinger investigated bounded distributive lattices K such that, for all bounded distributive lattices L and M in a given class, $K * L = K * M$ implies $L = M$ ([3]).

At the 1981 Banff Conference on Ordered Sets, F. Yaqub asked for a description of the largest class $\mathcal{L}$ of finite distributive lattices such that, for every non-trivial Boolean lattice B and every $L \in \mathcal{L}$, $B * L = B * L'$ implies $L = L'$ ([12], p. 849).

The problem as stated must be formulated more precisely, but we answer the question, as well as the corresponding one for isomorphisms (Theorems 2 and 3). Our tool is Priestley duality for distributive lattices.

2. Definitions, Notation, and Basic Results

For basic notions, see [6]. A poset is *connected* if, for all $p, q \in P$, there exist $n \in \mathbb{N}$ (which may be taken to be even) and $p_1, \ldots, p_n \in P$ such that $p = p_1 \leqslant p_2 \geqslant p_3 \leqslant \cdots \leqslant p_n = q$. A *component* of a poset is a non-empty maximal connected subset. A subset U of a poset P is an *up-set* if, for all $u \in U$ and $p \in P$, $u \leqslant p$ implies $p \in U$. A partially ordered topological space P is *totally order-disconnected* if, for all $p, q \in P$ such that $p \nleq q$, there exists a clopen up-set U such that $p \in U$ and $q \notin U$. A *Priestley space* is a compact totally order-disconnected space.

Given an ordered space P, let $D(P)$ denote the bounded distributive lattice of clopen up-sets. Given a bounded distributive lattice L, let $P(L)$ denote the Priestley space of prime filters, partially ordered by set-inclusion and with the topology generated by the subbasis

$$\{\{F \in P(L) \mid a \in F\}, \{F \in P(L) \mid a \notin F\} \mid a \in L\}.$$

The operators D and P extend to functors which yield a dual equivalence between the categories of bounded distributive lattices with $\{0, 1\}$-preserving homomorphisms and Priestley spaces with continuous order-preserving maps. For some consequences of Priestley duality, see [9] and [10].

It is well known that $P(L)$ is an antichain if L is a Boolean lattice ([6], Theorem 9.8). Also, $P(L)$ is a finite poset with the discrete topology if L is a finite distributive lattice. Further, $P(L \times M)$ is *order-homeomorphic* (order-isomorphic via a map that is a homeomorphism) to $P(L) + P(M)$, the disjoint sum of the ordered spaces ([6], Exercise 10.3(iii)). Lastly, $P(L * M)$ is order-homeomorphic to $P(L) \times P(M)$. It can be shown that, for bounded distributive lattices L and M, $L * M$ is isomorphic to $L^{P(M)}$, the lattice of continuous order-preserving maps from $P(M)$ to L with the discrete topology, ordered pointwise;

L is associated with the constant maps, and M with the maps $\mathbf{2}^{P(M)}$, where $\mathbf{2} = \{0, 1\}$ ([5], Theorem and Corollary, [4], Corollary 2.3, and [11], Theorem). In [14] and [15], this aspect of duality is used to understand *generalized Post algebras*, coproducts of Boolean lattices and bounded distributive lattices. (See Figures 1–4.)

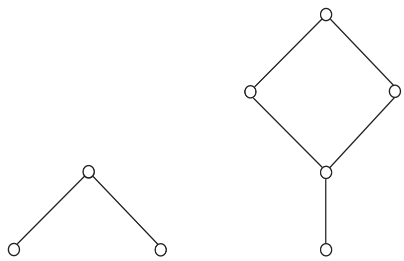

Figure 1 The poset P and the lattice $L = D(P)$

Figure 2 The poset Q and the lattice $M = D(Q)$

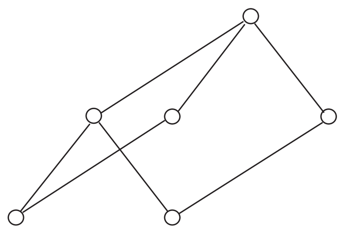

Figure 3 The poset $P \times Q$

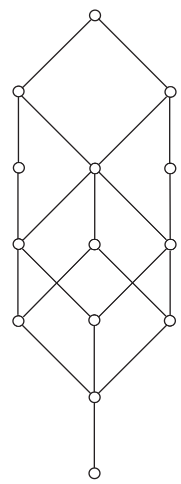

Figure 4 The lattice $L * M \cong D(P \times Q)$

3. The Solution to the Problem for Isomorphisms and Equality

We must emend Yaqub's problem because of the following example: Let B be a Boolean lattice such that $B^2 \cong B$. (For instance, let $B := \mathbf{2}^\omega$.) Then for any finite poset P, $B^P \cong B^{(2\times P)}$. Hence, for any non-trivial finite distributive lattice L, $B * L \cong B * L^2$, where $L^2 \ncong L$ by cardinality considerations. Hence, there exist bounded distributive lattices P and L' such that $P = B*L = B*L'$ but the image of L in P is not equal or even isomorphic to the image of L' in P.

Therefore, in the problem, we must restrict the class to which L' can belong. We insist that L and L' belong to the *same* class.

A class $\mathcal{L}$ of finite distributive lattices is *shabazz* if, for all lattices M and N, $M \times N^2 \in \mathcal{L}$ implies $M \times N \in \mathcal{L}$. A finite distributive lattice is *square-free* if it has no direct factor of the form N^2 (N a non-trivial lattice). Let $\mathcal{B}$ be the class of non-trivial Boolean lattices.

LEMMA 3.1. *Let X be an antichain, P and P' posets, C a component of P, and $\Psi : X \times P \cong X \times P'$ an order-isomorphism. Let $\pi_X : X \times P' \to X$ be the projection. Then, for all $x \in X$, there exists $x' \in X$ such that*

$$(\pi_X \circ \Psi)[\{x\} \times C] = \{x'\}.$$

Proof. Suppose that $\Psi(x, p_0) = (x', p_0')$ and $\Psi(x, p_1) = (x'', p_1')$. If $p_0 \leqslant p_1$, then $(x', p_0') \leqslant (x'', p_1')$, so that $x' = x''$. □

THEOREM 3.2. *Let $\mathcal{L}$ be a shabazz class of finite distributive lattices. The following are equivalent:*

(1) *for all $B \in \mathcal{B}$ and $L, L' \in \mathcal{L}$, $B * L \cong B * L'$ implies $L \cong L'$;*
(2) *every member of $\mathcal{L}$ is square-free.*

Proof. Assume $L \cong M \times N^2$ for some $L \in \mathcal{L}$ and lattices M and N (N non-trivial). Let $L' := M \times N$ and let $B := \mathbf{2}^\omega$. As $B \cong B^2$, we have $B * L \cong B * L'$.

Now assume $B \in \mathcal{B}$ and let L and L' be square-free finite distributive lattices. Then $B * L \cong B * L'$ implies there exists an order-isomorphism $\Psi : X \times P \cong X \times P'$ where $X := P(B)$, $P := P(L)$, and $P' := P(L')$; P and P' are finite posets with pairwise non-isomorphic components. Let $\pi_{P'} : X \times P' \to P'$ be the projection and fix $x \in X$. By Lemma 3.1, the map

$$p \mapsto (\pi_{P'} \circ \Psi)(x, p) \quad (p \in C)$$

is an order-isomorphism on each component C of P. □

THEOREM 3.3. *Let $\mathcal{L}$ be a class of finite distributive lattices. The following are equivalent:*

(1) *for all $B \in \mathcal{B}$ and $L, L' \in \mathcal{L}$, $B * L = B * L'$ implies $L = L'$;*
(2) *every member of $\mathcal{L}$ is rigid.*

Proof. Let L be a finite distributive lattice and $\phi : L \cong L$ a non- trivial automorphism. Define $\Phi : L^2 \to L^2$ by $\Phi(a, b) = (a, \phi(b))$ $(a, b \in L)$. Then Φ is an automorphism of $\mathbf{2}^2 * L$ that does not map constants to constants.

Now let X be a non-empty Priestley space that is an antichain. Let P and P' be finite rigid posets. Let $\Psi : X \times P \cong X \times P'$ be an order-homeomorphism. We must show that, for all $U \in D(P)$, there exists $U' \in D(P')$ such that $\Psi[X \times U] = X \times U'$.

Without loss of generality, P and P' are connected. By Lemma 3.1, for all $x \in X$, there exists $x' \in X$ such that $\Psi[\{x\} \times P] = \{x'\} \times P'$. For all $x \in X$ and $U \in D(P)$, let $U'_x \in D(P')$ be such that $\Psi[\{x\} \times U] = \{x'\} \times U'_x$. (Since $\{x\} \times U$ is an up-set of $X \times P$, $\Psi[\{x\} \times U]$ is an up-set of $X \times P'$, and we already know that it is of the form $\{x'\} \times V$ for some set $V \subseteq P'$. This V must be an up-set of P'.)

It suffices to show that, for all $x_0, x_1 \in X$ and $p \in P$,

$$(\pi_{P'} \circ \Psi)(x_0, p) = (\pi_{P'} \circ \Psi)(x_1, p).$$

By Lemma 3.1, the maps $p \mapsto (\pi_{P'} \circ \Psi)(x_0, p)$ and $p \mapsto (\pi_{P'} \circ \Psi)(x_1, p) (p \in P)$ are order-isomorphisms, so, by rigidity, they are equal. □

COROLLARY 3.4. *The largest class of finite distributive lattices with the property that, for any two lattices L, L' in the class and any non-trivial Boolean lattice B,*

$$B * L = B * L' \quad \textit{implies} \quad L = L',$$

is the class of all finite rigid distributive lattices.

Hence we have solved the emended form of Yaqub's problem, where we insist that the lattices L and L' belong to the same class.

4. Related Results

In [3], a research announcement of results that apparently were never published*, it is stated that if K, L, and M are bounded distributive lattices with L and M rigid, then $K * L = K * M$ implies $L = M$. (The author discovered Theorem 3.3 independently of [3].) The interpretation of this statement is *not* the same as in our paper, as is evidenced by the case where K, L, and M are each the three-element chain (cf. [8], Figure 4).

In [2], Theorem 2.4(i), it is shown that there exist countable Boolean algebras L and L' such that $\mathbf{2}^2 * L \cong \mathbf{2}^2 * L'$ but $L \ncong L'$. It also follows form (1.4) in [2] that, for non-trivial finite distributive lattices L, L', and M, $L * M \cong L' * M$ implies $L \cong L'$.

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*The author was unsuccessful in his attempt to obtain a copy of the (unpublished) paper announced in [3].

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