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Strictly Order-Preserving Maps into $\mathbb{Z}$, II. A 1979 Problem of Erné

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Abstract. A lattice L is constructed with the property that every interval has finite height, but there exists no strictly order-preserving map from L to $\mathbb{Z}$. A 1979 problem of Erné (posed at the 1981 Banff Conference on Ordered Sets) is thus solved. It is also shown that if a poset P has no uncountable antichains, then it admits a strictly order-preserving map into $\mathbb{Z}$ if and only if every interval has finite height.

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1. Introduction

When does a poset admit a strictly order-preserving map into the chain of integers, $\mathbb{Z}$?

Let P be a poset. A function $f\colon P \to \mathbb{Z}$ is *strictly order-preserving* if, for all $p, q \in P$, $p < q$ implies $f(p) < f(q)$ (see Figures 1 and 2).

Clearly every finite poset admits such a map; but the chain $\mathbb{N} \cup \{\infty\}$ (the natural numbers with a top element adjoined) does not.

An obvious necessary condition for the existence of such a function is that every interval must have finite height. (If $x, y \in P$, the set

$$[x, y] = \{z \in P \mid x \leq z \leq y\}$$

is an *interval*; its *height* is the supremum of the lengths of the chains in the interval.) It is known that if P is countable, then this necessary condition is also sufficient:

PROPOSITION ([2], Theorem 1.9.10). *A countable poset admits a strictly order-preserving map into $\mathbb{Z}$ if and only if every interval has finite height.*

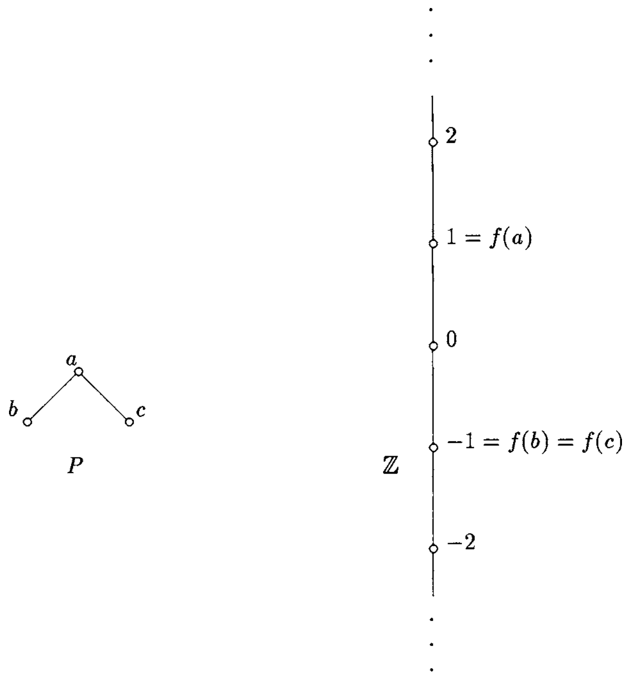

Figure 1. A strictly order-preserving map $f\colon P \to \mathbb{Z}$.

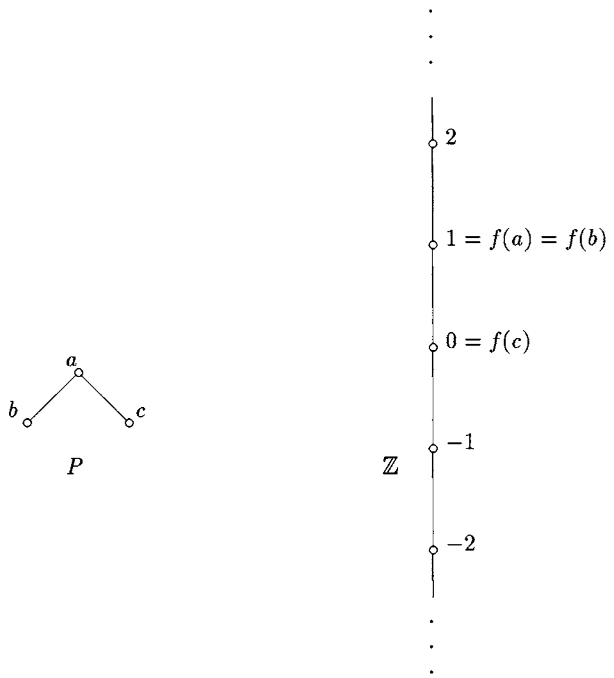

Figure 2. An order-preserving map $f\colon P \to \mathbb{Z}$ that is not strictly order-preserving.

This condition is not sufficient in general: Erné has constructed a (necessarily uncountable) poset such that every interval has finite height, but there exists no strictly order-preserving map from the poset into $\mathbb{Z}$ ([2], Example 1.9.11). In 1979, Erné wrote ([2], p. 273), "We do not know whether in lattices the countability condition can be dropped." Erné then posed the following problem at the 1981 Banff Conference on Ordered Sets ([5], p. 843):

PROBLEM (Erné, 1979/1981). *Consider the following two properties of a poset P:*

(1) *There exists a strictly order-preserving map from P to $\mathbb{Z}$;*
(2) *every interval of P has finite height.*

When does (2) *imply* (1)*?*
In particular, is this true for lattices?

We answer the second question in the negative (Proposition 2.2). We also prove that (1) and (2) are equivalent for posets with no uncountable antichains (Proposition 3.2).

In Part I of this paper, the first author solved a problem of Daykin posed at the 1984 Banff Conference on Graphs and Order (see [1], Problem 8.1 and [6], pp. 532–533): If P is a countable poset, S a subset, and $g\colon S \to \mathbb{Z}$ a strictly order-preserving map, when can g be extended to a strictly order-preserving map $\Psi\colon P \to \mathbb{Z}$?

For results about maps onto $\mathbb{N}$, see [4].

2. The Solution to Erné's Problem

First we construct a poset that does not admit a strictly order-preserving map into $\mathbb{Z}$, even though every interval has finite height:

Let A be the set of natural numbers ordered as an antichain; let B be the set of all functions $b\colon \mathbb{N} \to \mathbb{N}$ ordered as an antichain. Between $a \in A$ and $b \in B$, let there be a chain $c_1, c_2, \ldots, c_{b(a)}$ with $b(a)$ elements, so that $a > c_1 > \cdots > c_{b(a)} > b$. Let C be the set of all these chains and let Q be the poset $A \cup B \cup C$ with no additional comparability relations (Figure 3).

LEMMA 2.1. *There exists no strictly order-preserving map $f\colon Q \to \mathbb{Z}$.*

Proof. Suppose, for a contradiction, that there exists such a map f. Let $F(n) := \max\{n, f(1), f(2), \ldots, f(n)\}$ for all $n \in \mathbb{N}$. Note that, for every function $b\colon \mathbb{N} \to \mathbb{N}$ and every $n \in \mathbb{N}$,

$$F(n) - f(b) > b(n)$$

because $f\colon Q \to \mathbb{Z}$ is strictly order-preserving. Let $B\colon \mathbb{N} \to \mathbb{N}$ be the function

$$B(n) = [F(n)]^2 \quad (n \in \mathbb{N}).$$

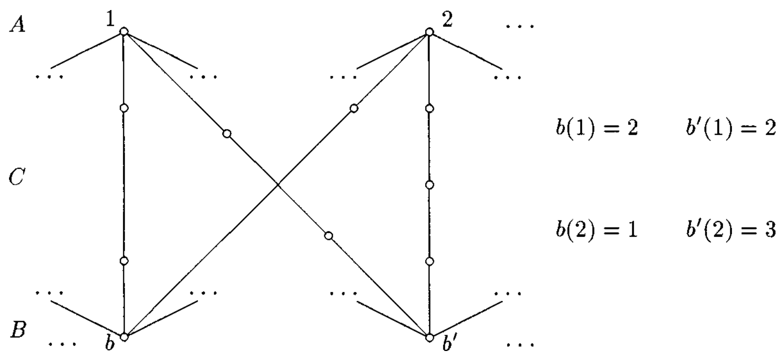

Figure 3. Part of the poset Q.

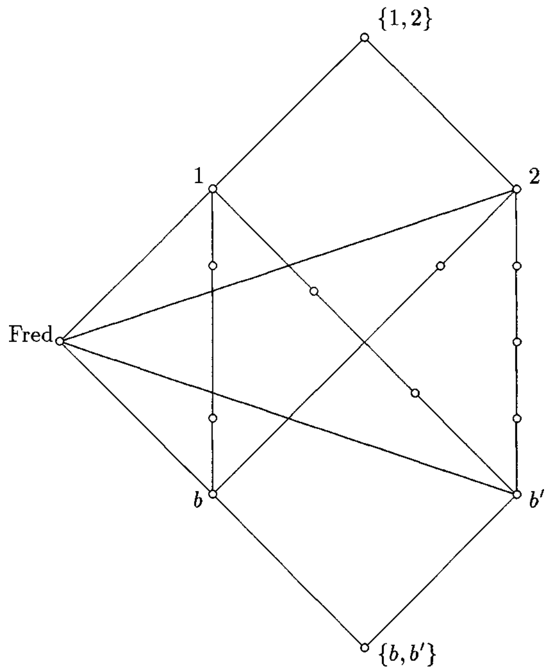

Figure 4. Part of the poset L.

Then, for $k = f(B) \in \mathbb{N}$ and all $n \in \mathbb{N}$,

$$F(n) - k > [F(n)]^2,$$

which is absurd, since $F(n)$ tends to infinity as n does. $\square$

Let A^* be the poset of finite subsets of A and let B^* be the dual of the poset of finite subsets of B; identify A and B with the singletons of A^* and B^*, respectively.

Let $L := A^* \cup B^* \cup C \cup \{\text{Fred}\}$, where $a > \text{Fred} > b$ for all $a \in A$ and $b \in B$ (Figure 4).

PROPOSITION 2.2. *The poset L is a lattice such that every interval has finite height, but it does not admit a strictly order-preserving map into $\mathbb{Z}$.*

Proof. Since L contains Q, there exists no strictly order-preserving map from L to $\mathbb{Z}$ by Lemma 2.1. By construction every interval has finite height. An easy check shows that L is a lattice. □

The above proposition answers Erné's 1979 question in the negative.

3. Posets with No Uncountable Antichains

In this section we extend Erné's observation (Section 1, Proposition). (For results pertaining to posets with no uncountable antichains, see, for instance, [3].)

LEMMA 3.1. *Let P be a poset with no uncountable antichains. If every interval has finite height, then P is countable.*

Proof. Let A be a maximal antichain; it is countable. Fix $a \in A$. For every $n \in \mathbb{N}$, there exist only countably many $p \in P$ such that the height of the interval $[a, p]$ is n. Hence, the set of elements comparable to a is countable. Thus P is a countable union of countable sets. □

PROPOSITION 3.2. *A poset with no uncountable antichains admits a strictly order-preserving map into $\mathbb{Z}$ if and only if every interval has finite height.*

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