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The automorphism group of a function lattice: A problem of Jónsson and McKenzie

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Abstract. It is shown that $\text{Aut}(L^Q)$ is naturally isomorphic to

$$\text{Aut}(L) \times \text{Aut}(Q)$$

when L is a directly and exponentially indecomposable lattice, Q a non-empty connected poset, and one of the following holds: Q is arbitrary but L is a *jm*-lattice, Q is finitely factorable and L is complete with a join-dense subset of completely join-irreducible elements, or L is arbitrary but Q is finite. A problem of Jónsson and McKenzie is thereby solved. Sharp conditions are found guaranteeing the injectivity of the natural map $v_{P,Q}$ from $\text{Aut}(P) \times \text{Aut}(Q)$ to $\text{Aut}(P^Q)$ (P and Q posets), correcting misstatements made by previous authors. It is proven that, for a bounded poset P and arbitrary Q, the Dedekind-MacNeille completion of P^Q, $\mathbf{DM}(P^Q)$, is isomorphic to $\mathbf{DM}(P)^Q$. This isomorphism is used to prove that the natural map $v_{P,Q}$ is an isomorphism if $v_{\mathbf{DM}(P),Q}$ is, reducing a poset problem to a more tractable lattice problem.

1. Introduction

The *function space* P^Q, where P and Q are ordered sets, is the poset of order-preserving maps from Q to P ordered pointwise. If L is a lattice, L^Q is a *function lattice.* A poset R is *directly indecomposable* if $P \times Q \cong R$ implies precisely one of P and Q is trivial. It is *exponentially indecomposable* if $P^Q \cong R$ implies Q is trivial. The automorphism group of R is denoted $\text{Aut}(R)$. Jónsson and McKenzie ask in [19, Problem 12.3] if

$$\text{Aut}(L^Q) \cong \text{Aut}(L) \times \text{Aut}(Q)$$

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when L is a directly and exponentially indecomposable lattice, and Q a non-empty connected poset, which, possibly, is finite. We are working towards the most general conditions for which the answer is yes.

There is in general a trade-off between conditions on the base and conditions on the exponent. In §5 (Theorem 5.6) we handle the case of arbitrary exponent (subject to the other hypotheses), but we assume the base is a *jm*-lattice (a complete lattice whose completely join-irreducible elements form a join-dense subset, and dually for meet). We vastly generalize Duffus and Wille's result for finite exponents and bases of finite length ([14, Theorem 1]), as every such base is a *jm*-lattice. They use the "scaffolding," a representation valid only for lattices of finite length analogous to Priestley's for distributive lattices ([24]), so their technique is inherently limited. We extend Markowsky's representation for *jm*-lattices by bipartite directed graphs ([22, Definition 1.3(d)]) to obtain what Duffus and Rival call a "logarithmic property" ([13, §1]).

In §7, we handle the case of arbitrary base, but we assume the exponent is finite (Theorem 7.7). We at any rate solve the problem above posed by Jónsson and McKenzie. We generalize their own theorem which only takes care of subdirectly irreducible lattices ([19, Theorem 11.5]).

Our trick is to take the ideal completion of L^Q repeatedly (see [20, §I]). By [12, Theorem 3.1], the operation of completing by ideals commutes with exponentiating by Q, so we eventually get the dual of an algebraic lattice, which therefore has a join-dense subset of completely join-irreducible elements ([4, Theorem 8.8.16]). In other words, we get a hybrid of the function lattices dealt with in §§5 and 7. The base is a j-lattice, "half" of a *jm*-lattice: it is complete and the completely join-irreducible elements are join-dense. It turns out that, for such a base, the exponent need not be finite, but merely *finitely factorable*, a product of finitely many directly indecomposable posets (Theorem 6.18). [Although independently arrived at, our method resembles Novotný's ([23, §§7 and 8]) and Bauer's ([1, §5]). The former analyzed function lattices with totally ordered bases. The latter looked at a variant of the above problem but assumed that the *ideal* lattice of the base was directly and exponentially indecomposable ([1, Satz 6.5.3]).]

The fact we can expand the class of exponents if we restrict the class of bases illustrates the trade-off mentioned earlier, in more ways than one: Jónsson ([18, Theorem 14]) has shown that we may replace the complete lattice by a bounded poset (yielding a j-poset), *provided* we assume the finitely factorable exponent satisfies the ascending chain condition. Indeed, Jónsson proves that the "natural" map

$$\nu_{P,Q}\colon \mathrm{Aut}(P) \times \mathrm{Aut}(Q) \to \mathrm{Aut}(P^Q)$$

given by

$$[\nu_{P,Q}(\lambda, \rho)](f) = \lambda \circ f \circ \rho^{-1} \qquad \text{for all } \lambda \in \mathrm{Aut}(P),\ \rho \in \mathrm{Aut}(Q), \text{ and } f \in P^Q$$

is an isomorphism. He uses a lemma ([18, Lemma 13]) which is perfectly valid in the context of the paper, but nevertheless requires some minor additional hypotheses. It might be tempting to think that $\nu_{P,Q}$ is a group-embedding for arbitrary P and Q (see [8, p. 44 and §5], [11, §5], [18, §1], and [19, §11]). In §4 we show it is not so. We provide the conditions (which cannot be relaxed) under which $\nu_{P,Q}$ is necessarily an embedding, in fact, an embedding of ordered groups from $\mathrm{Aut}(P) \times \mathrm{Aut}(Q)^\partial$ to $\mathrm{Aut}(P^Q)$ (where P^∂ is the dual of P). See Theorem 4.2.

Our results for function lattices carry over to bounded function spaces, provided the Dedekind-MacNeille completion of P (rather than P) satisfies the indecomposability requirements. As the completion is obviously a complete lattice, we can pull down our results for complete lattices. In §8, we consider $\mathrm{Aut}(P^Q)$ when the base is bounded (but does not necessarily satisfy any other conditions) and Q is arbitrary. We prove that $\mathbf{DM}(P^Q) \cong \mathbf{DM}(P)^Q$ (Theorem 8.6). We deduce that if $\nu_{\mathbf{DM}(P),Q}$ is surjective, so is $\nu_{P,Q}$ (Theorem 8.9). [In fact, we deduce a slightly sharper result (Proposition 8.8).]

In Table 1 we list the various extra conditions on the base L and exponent Q for which it is known that $\mathrm{Aut}(L^Q) \cong \mathrm{Aut}(L) \times \mathrm{Aut}(Q)$, along with the papers containing the corresponding result. The row headers in slanted type are the conditions on the base; the column headers in bold type are conditions on the exponent. For

Table 1

base vs. **Exponent**	**Finite**	**Fin. Factorable & ACC**
subdirectly irreducible	Theorem 7.7, [19]	
finite length	Theorems 5.6, 7.7, [14]	Theorem 5.6
jm-lattice	Theorems 5.6, 7.7	Theorem 5.6
j-lattice	Theorems 6.18, 7.7, [18]	Theorem 6.18, [18]
j-poset	[18]	[18]
arbitrary	Theorem 7.7	

base vs. **Exponent**	**Finitely Factorable**	**Arbitrary**
subdirectly irreducible		
finite length	Theorem 5.6	Theorem 5.6
jm-lattice	Theorem 5.6	Theorem 5.6
j-lattice	Theorem 6.18	
j-poset		
arbitrary		

instance, that the isomorphism holds when the base is a j-lattice and the exponent a finitely factorable poset with the ascending chain condition (satisfying the other conditions) is a corollary of Theorem 6.18 as well as a theorem in [18]. Except for the row for j-posets, all the bases are lattices.

2. General definitions and notation

Our definitions and notation come largely from [9]. In the sequel, results within a section will be referred to in that section without a section number.

Let P and Q be ordered sets. Denote by P^Q the poset of order-preserving maps from Q to P ordered pointwise. That is, for $f, g \in P^Q$, $f \leq g$ in P^Q if $f(q) \leq g(q)$ for all $q \in Q$. If $p \in P$, let $\bar{p}$ denote the constant map whose sole value is p.

Denote by $P + Q$ the disjoint sum of posets P and Q.

Let $\mathscr{P}$ denote the class of all posets. Let $\mathscr{P}_{\text{fin}}$ denote the class of all finite posets. A singleton poset is called *trivial.* Note that we are allowing posets to be empty.

Let R be a poset and $\mathscr{Q}$ a class of posets. We call R *exponentially indecomposable with respect to* $\mathscr{Q}$ if, for $P \in \mathscr{P}$ and $Q \in \mathscr{Q}$, $R \cong P^Q$ implies Q is trivial. If $\mathscr{Q} = \mathscr{P}$, we simply call R *exponentially indecomposable.*

We call a poset R *directly indecomposable* if, for all posets P and Q, $R \cong P \times Q$ implies exactly one of P and Q is trivial.

A poset is *finitely factorable* if it is a finite product of directly indecomposable posets. (The trivial poset is finitely factorable; the empty poset is not.)

We call a poset R *connected* if, for all posets P and Q, $R \cong P + Q$ implies P or Q is empty. Equivalently, for all $r, s \in R$, there is a natural number n (which, without loss of generality, may be chosen to be odd) and there are elements $r_0, \ldots, r_n \in R$ such that $r = r_0 \leq r_1 \geq r_2 \leq \cdots \leq r_n = s$.

If S, T, and U are sets, and $f\colon S \to T$, $g\colon T \to U$ functions,

$$g \circ f\colon S \to U$$

is the composition of f and g. If $R \subseteq S$, then $f[R] := \{f(r) \mid r \in R\} \subseteq T$, $\operatorname{Im} f := f[S]$, and $f \restriction R$ is the restriction of f to R. The inclusion map is denoted $\iota_R\colon R \to S$. The identity on S is denoted id_S or $\mathrm{id}(S)$.

If P and Q are posets, $f\colon P \to Q$ is an *order-embedding* if for all $p, p' \in P$, $p \leq p'$ if and only if $f(p) \leq f(p')$. If it is also onto, it is an *order-isomorphism.* The set of automorphisms of P is denoted $\operatorname{Aut}(P)$.

The dual of a poset P is denoted P^∂.

If P and Q are posets, the natural map

$$\nu_{P,Q}\colon \operatorname{Aut}(P) \times \operatorname{Aut}(Q)^\partial \to \operatorname{Aut}(P^Q)$$

is defined, for all $\lambda \in \mathrm{Aut}(P)$, $\rho \in \mathrm{Aut}(Q)$, and $f \in P^Q$, by

$$[\nu_{P,Q}(\lambda, \rho)](f) = \lambda \circ f \circ \rho^{-1}.$$

The least element of a poset P (if it exists) is denoted 0_P or 0, the greatest element by 1_P or 1. A poset with 0 and 1 is *bounded.*

If P is a poset, $Q \subseteq P$, and $p \in P$, let $\downarrow_Q p := \{q \in Q \mid q \leq p\}$. Let $\downarrow p := \downarrow_P p$. A subset $D \subseteq P$ is a *down-set* if $\downarrow d \subseteq D$ for all $d \in D$. A down-set in P^∂ is an *up-set* of P.

Let P be a poset. An element $p \in P$ is *completely join-irreducible* if, for all $S \subseteq P$ such that $\bigvee S$ exists, $p = \bigvee S$ implies $p \in S$. The set of all completely join-irreducible elements of P is denoted $\mathcal{J}_c(P)$. Dually, we define *completely meet-irreducible elements* and $\mathcal{M}_c(P)$. It is easy to see that this definition is equivalent to [19, Definition 6.1].

The notation defined below is adapted from (and, one may prove, consistent with) [19, Definition 6.2(iii)].

Let P be a poset with 1, $S \subseteq P$. Let

$$\nabla S := \{\bigvee T \mid T \subseteq S \text{ and } \bigvee T \text{ exists}\}.$$

If $\nabla S = P$, then S is *join-dense in P.*

Dually, we define ΔS for subsets of posets with 0; if $\Delta S = P$, S is *meet-dense in P.*

The notation below derives from [19, Definition 6.4].

Let P be a bounded poset and Q any poset. For all $p \in P$ and $q \in Q$, let $j(p, q)\colon Q \to P$ be the function such that, for all $t \in Q$,

$$[j(p, q)](t) = \begin{cases} p & \text{if } t \geq q, \\ 0 & \text{else.} \end{cases}$$

Dually, $k(p, q)\colon Q \to P$ is the function such that, for all $t \in Q$,

$$[k(p, q)](t) = \begin{cases} p & \text{if } t \leq q, \\ 1 & \text{else.} \end{cases}$$

3. The logarithmic approach

The idea behind the logarithmic approach is to turn a problem about powers into a (hopefully easier) problem about products. A logarithm ([13, §1], [8, §3], [17, §7]) turns powers to products and products to sums. In particular, it should be the case that P is directly indecomposable if and only if "log P" is connected, and P is exponentially indecomposable if and only if "log P" is directly indecomposable.

We have quite satisfactory theorems regarding automorphism groups of powers of *connected* posets. Hashimoto's argument for [16, Theorem 1] establishes the *strict refinement property* (cf. [5, Definition 4.2]) for products of connected posets. (Essentially this observation is made in [19, §2].)

HASHIMOTO'S REFINEMENT THEOREM. *If P, Q, R, and S are connected posets, and*

$$\phi: P \times Q \cong R \times S,$$

then there are posets T, U, V, and W, and maps

$$\alpha: T \times U \cong P, \qquad \beta: V \times W \cong Q, \qquad \gamma: T \times V \cong R, \quad \text{and} \quad \delta: U \times W \cong S$$

such that, for all $t \in T$, $u \in U$, $v \in V$, and $w \in W$,

$$\phi(\alpha(t, u), \beta(v, w)) = (\gamma(t, v), \delta(u, w)).$$

Using this property, Duffus proved that any automorphism of a product of relatively prime connected posets is a product of automorphisms of the factors ([10, Corollary 2(a)]).

Hence if P is directly indecomposable and Q connected, "$\log(P^Q)$," isomorphic to $Q \times$ "log P," should be a product of connected structures. If P is in addition exponentially indecomposable, "log P" should be directly indecomposable and relatively prime to Q. Hence, assuming Aut ("log P") $\cong$ Aut P, we should be able to employ an analogue of Duffus's result to show that $\mathrm{Aut}(P^Q) \cong \mathrm{Aut}\, P \times \mathrm{Aut}\, Q$.

As an example, we want to use Priestley duality to obtain a logarithm for distributive lattices with properties almost like the desired ones. Let us remind ourselves of the rudiments of this duality. A *Priestley space* P is a compact totally order-disconnected ordered space, that is, if $p, q \in P$ and $p \not\leq q$, then there is a clopen down-set D such that $p \notin D$, $q \in D$. See [9, 10.15, 10.25, and 10.26] and [24, Theorems 1–3]. The category $\mathbf{D}$ of bounded distributive lattices with bound-preserving homomorphisms is dually equivalent to the category $\mathbf{P}$ of Priestley spaces with continuous order-preserving maps. The functor D from $\mathbf{D}$ to $\mathbf{P}$ assigns to each bounded distributive lattice its poset of prime filters with an appropriate topology. The functor E from $\mathbf{P}$ to $\mathbf{D}$ assigns to each Priestley space its lattice of clopen up-sets. If $L_1, L_2 \in \mathbf{D}$ and $f: L_1 \to L_2$ is a bound-preserving homomorphism, then $D(f): D(L_2) \to D(L_1)$ maps each prime filter F of L_2 to $f^{-1}(F)$. If $P_1, P_2 \in \mathbf{P}$ and

$g: P_1 \to P_2$ is a continuous order-preserving map, then $E(g): E(P_2) \to E(P_1)$ maps each clopen up-set U of P_2 to $g^{-1}(U)$.

As dual equivalences preserve and reflect isomorphisms, if $L \in \mathbf{D}$ and $f \in \mathrm{Aut}(L)$, then $D(f)^{-1}$ [which equals $D(f^{-1})$] belongs to $\mathrm{Aut}(D(L))$, the group of order-homeomorphisms on $D(L)$. If $f_1, f_2 \in \mathrm{Aut}(L)$, then

$$D(f_1 \circ f_2)^{-1} = (D(f_2) \circ D(f_1))^{-1} = D(f_1)^{-1} \circ D(f_2)^{-1},$$

so the map from $\mathrm{Aut}(L)$ to $\mathrm{Aut}(D(L))$ defined by

$$f \mapsto D(f)^{-1} \qquad (f \in \mathrm{Aut}(L))$$

is a group-isomorphism.

Let $L \in \mathbf{D}$ and $Q \in \mathbf{P}$. The set of continuous order-preserving maps from Q to L (with the discrete topology) is a bounded distributive lattice. If Q is finite, its topology is discrete, so this lattice equals L^Q. Davey proved its Priestley dual space is order-homeomorphic to $Q \times D(L)$ ([7, Theorem and Corollary]; see also [6, Corollaries 2.3 and 2.12]).

Now assume L is directly and exponentially indecomposable, and Q is finite, non-empty, and connected. By the above,

$$\mathrm{Aut}(L^Q) \cong \mathrm{Aut}(D(L^Q)) \cong \mathrm{Aut}(Q \times D(L)).$$

The Priestley spaces $D(L)$ and Q are relatively prime in the category of ordered spaces. For if F is a common factor, then it is finite, in $\mathbf{P}$, and

$$D(L) \cong F \times G$$

for some $G \in \mathbf{P}$. By Davey's result and the finiteness of F,

$$L \cong ED(L) \cong E(F \times G) \cong (E(G))^F,$$

which implies F is trivial, because L is exponentially indecomposable.

To complete the proof, we need a lemma which may be drawn from [2, 3.2 and 3.3].

Let P, Q, R, S, and X be ordered topological spaces such that the graph of the order relation of each space is closed in the square. Assume Q and S are finite, X is compact,

$$X \cong P \times Q,$$

and there is no continuous order-preserving map from X onto the two-element antichain $\bar{\mathbf{2}}$ *(a totally unordered poset with the discrete topology).*

If $(p, q) \in P \times Q$ *and*

$$\phi: P \times Q \cong R \times S,$$

then there exist subspaces $T \subseteq R$ *and* $U \subseteq S$ *such that*

$$\phi[P \times \{q\}] = T \times U$$

and U is a direct factor of S.

We may now prove an analogue of Duffus's result for Priestley spaces.

Let P and $Q \in \mathbf{P}$ *where Q is finite, connected, and relatively prime to P in the category of ordered spaces. Assume there is no continuous order-preserving map from P onto* $\bar{\mathbf{2}}$ *(with the discrete topology). Then for every* $\phi \in \mathrm{Aut}(P \times Q)$ *there exist* $\lambda \in \mathrm{Aut}(P)$ *and* $\rho \in \mathrm{Aut}(Q)$ *such that* $\phi = \lambda \times \rho$.

Proof. Without loss of generality, $P, Q \neq \emptyset$. There is no continuous order-preserving map from $P \times Q$ onto $\bar{\mathbf{2}}$. For each $q \in Q$,

$$\phi[P \times \{q\}] = \lambda_q[P] \times \{\rho(q)\}$$

for some $\lambda_q \in \mathrm{Aut}(P)$ and $\rho \in \mathrm{Aut}(Q)$.

If $q, r \in Q$ and $q \leq r$, then for all $p \in P$

$$(\lambda_q(p), \rho(q)) \leq (\lambda_r(p), \rho(q)) \leq (\lambda_r(p), \rho(r))$$

so that

$$(p, q) \leq \phi^{-1}(\lambda_r(p), \rho(q)) \leq (p, r).$$

Hence

$$\phi^{-1}(\lambda_r(p), \rho(q)) = (p, q)$$

so $\lambda_q(p) = \lambda_r(p)$ for all $p \in P$. As Q is connected, there exists $\lambda \in \mathrm{Aut}(P)$ such that $\lambda = \lambda_q$ for all $q \in Q$. Hence $\phi = \lambda \times \rho$. □

As expected, $L \in \mathbf{D}$ is decomposable if and only if there is a continuous order-preserving map from $D(L)$ onto $\bar{\mathbf{2}}$. (Cf. [9, Exercise 10.3].) Therefore

$$\mathrm{Aut}(D(L) \times Q) \cong \mathrm{Aut}(D(L)) \times \mathrm{Aut}(Q).$$

Putting it all together, we get

$$\mathrm{Aut}(L^Q) \cong \mathrm{Aut}(L) \times \mathrm{Aut}(Q).$$

The case just considered points out the importance of having not just a logarithm but also a corresponding exponential. Lemma 6.5(iii) of [19] implies that, if L is a complete lattice and Q is any poset, $\mathscr{J}_c(L^Q)$ equals

$$\{j(k, q) \mid k \in \mathscr{J}_c(L), q \in Q\}$$

and is isomorphic to $Q^\partial \times \mathscr{J}_c(L)$, so $\mathscr{J}_c(-)$ seems to act like a logarithm. We now show that the operator ∇ acts as the corresponding exponential.

The first lemma is straightforward.

LEMMA 3.1. *Let L be a complete lattice, Q a poset, and $P \subseteq Q$. Let*

$$\Phi: L^Q \to L^P$$

be the restriction map

$$f \mapsto f\restriction_P \qquad (f \in L^Q),$$

and $\Psi: L^P \to L^Q$ the map defined by

$$[\Psi(f)](q) = \bigvee f[\downarrow_P q]$$

for all $f \in L^P$, $q \in Q$.

Then Φ and Ψ are order-preserving functions such that $\Phi \circ \Psi = \mathrm{id}(L^P)$ and $\Psi \circ \Phi \leq \mathrm{id}(L^Q)$. □

For the remainder of this section, let L be a complete lattice, Q a poset, $K \subseteq \mathscr{J}_c(L)$, and $P \subseteq Q$. Let

$$M := \nabla\{j(k, p) \mid k \in K, p \in P\} \subseteq L^Q$$

and $N := \nabla K \subseteq L$. For each $f \in M$ and $p \in P$, let

$$\mathscr{C}_p(f) := \{j(k, p) \mid f(p) \geq k \in K\}.$$

As $f \in M$, $f = \bigvee_{p \in P} (\bigvee \mathscr{C}_p(f))$. Hence, for all $q \in Q$,

$$f(q) = \bigvee_{p \in P} \bigvee_{g \in \mathscr{C}_p(f)} g(q).$$

The next two claims are easy.

LEMMA 3.2. *If $f \in M$, $p \in P$, $q \in Q$, but $p \nleq q$, then*

$$\bigvee_{g \in \mathscr{C}_p(f)} g(q) = 0. \qquad \square$$

LEMMA 3.3. *If $f \in M$, $k \in K$, $p, q \in P$, $p \leq q$, and $j(k, p) \in \mathscr{C}_p(f)$, then $j(k, q) \in \mathscr{C}_q(f)$.* $\square$

LEMMA 3.4. *For all $f \in M$ and $q \in P$,*

$$\bigvee_{p \in P} \bigvee_{g \in \mathscr{C}_p(f)} g(q) = \bigvee_{g \in \mathscr{C}_q(f)} g(q).$$

Proof. By Lemmas 2 and 3,

$$\begin{aligned}\bigvee_{p \in P} \bigvee_{g \in \mathscr{C}_p(f)} g(q) &= \Bigg[\bigvee_{\substack{p \in P \\ p \nleq q}} \bigvee_{g \in \mathscr{C}_p(f)} g(q)\Bigg] \vee \Bigg[\bigvee_{\substack{p \in P \\ p \leq q}} \bigvee_{g \in \mathscr{C}_p(f)} g(q)\Bigg] \\ &= \bigvee_{\substack{p \in P \\ p \leq q}} \bigvee_{g \in \mathscr{C}_p(f)} g(q) \\ &= \bigvee_{g \in \mathscr{C}_q(f)} g(q). \qquad \square\end{aligned}$$

We deduce the following claims.

LEMMA 3.5. *For all $f \in M$ and $p \in P$,*

$$f(p) = \bigvee_{g \in \mathscr{C}_p(f)} g(p). \qquad \square$$

LEMMA 3.6 *For all $f \in M$ and $q \in Q$,*

$$f(q) = \bigvee f[\downarrow_P q].$$

Proof. By Lemma 2,

$$f(q) = \bigvee_{\substack{p \in P \\ p \leq q}} \bigvee_{g \in \mathscr{C}_p(f)} g(q)$$

$$= \bigvee_{\substack{p \in P \\ p \leq q}} \bigvee_{g \in \mathscr{C}_p(f)} g(p).$$

By Lemma 5,

$$f(q) = \bigvee_{\substack{p \in P \\ p \leq q}} f(p). \qquad \square$$

From Lemmas 5 and 6, we get the following.

LEMMA 3.7. *For all $f \in M$ and $q \in Q$, $f(q) \in N$.* $\square$

PROPOSITION 3.8. *Let L be a complete lattice, Q a poset, $K \subseteq \mathscr{J}_c(L)$, and $P \subseteq Q$. Let*

$$M := \nabla \{j(k, p) \mid k \in K, p \in P\} \subseteq L^Q$$

and $N := \nabla K \subseteq L$. Then $M \cong N^P$ via the restriction map $f \mapsto f\restriction_P (f \in M)$.

Proof. Define Φ and Ψ as in Lemma 1. Let $\Phi^* := \Phi\restriction_M$ and $\Psi^* := \Psi\restriction_{N^P}$. By Lemma 7, Φ^* maps M to N^P, and, by Lemma 6, $\Psi^* \circ \Phi^* = \mathrm{id}_M$.

Now we show that, for all $h \in N^P$, $\Psi^*(h) \in M$. For each $h \in N^P$ and $p \in P$, let $\mathscr{D}_p(h) \subseteq K$ be such that $h(p) = \bigvee \mathscr{D}_p(h)$. Hence for all $q \in Q$

$$[\Psi^*(h)](q) = \bigvee h[\downarrow_P q]$$

$$= \bigvee_{\substack{p \in P \\ p \leq q}} \left(\bigvee \mathscr{D}_p(h) \right)$$

$$= \bigvee_{\substack{p \in P \\ p \leq q}} \bigvee_{k \in \mathscr{D}_p(h)} [j(k, p)](q)$$

so

$$\Psi^*(h) = \bigvee_{\substack{p \in P \\ p \leq q}} \bigvee_{k \in \mathscr{D}_p(h)} j(k, p),$$

and, hence, $\Psi^*(h) \in M$.

Thus, Ψ^* maps N^P to M; by Lemma 1, $\Phi^* \circ \Psi^* = \mathrm{id}(N^P)$. Therefore, $\Phi^*: M \cong N^P$. $\square$

4. The injectivity of the natural map

As stated in §1, it has been claimed that $\nu_{P,Q}$ (P, Q posets) is always a group-embedding. On the basis of this claim, the following result was asserted ([18, Lemma 13]).

For any posets A, C, and D, if the natural maps

$$\mathrm{Aut}(A) \times \mathrm{Aut}(C) \to \mathrm{Aut}(A^C)$$

$$\mathrm{Aut}(A^C) \times \mathrm{Aut}(D) \to \mathrm{Aut}[(A^C)^D]$$

are isomorphisms, then so is the natural map

$$\mathrm{Aut}(A) \times \mathrm{Aut}(C \times D) \to \mathrm{Aut}(A^{C \times D}).$$

We construct a counter-example (cf. [3, §17]). Let $A = \{x, y\}$ be a two-element antichain, and C and D two-element chains. Then $A^C = \{\bar{x}, \bar{y}\}$. Hence $\mathrm{Aut}(A)$ contains just id_A and the transposition $(x\,y)$, $\mathrm{Aut}(C) = \{\mathrm{id}_C\}$, and $\mathrm{Aut}(A^C) = \{\mathrm{id}(A^C), (\bar{x}\,\bar{y})\}$. Clearly

$$\nu_{A,C}(\mathrm{id}_A, \mathrm{id}_C) = \mathrm{id}(A^C) \quad \text{and} \quad \nu_{A,C}((x\,y), \mathrm{id}_C) = (\bar{x}\,\bar{y}),$$

so $\nu_{A,C}$ is an isomorphism. Similarly, $\nu_{A^C,D}$ is an isomorphism. On the other hand, $\nu_{A,C \times D}$ cannot be, for

$$\mathrm{Aut}(A) \times \mathrm{Aut}(C \times D)$$

has four elements, but $\mathrm{Aut}(A^{C \times D})$ only two.

Lemma 13 of [18] is correct when the base is not an antichain and the exponents are non-empty, for, in this case, ν is an embedding. In fact, it is an embedding of ordered groups. [For P a poset, $\mathrm{Aut}(P)$, ordered as a subset of P^P and made a group under composition, is an ordered group (*po-group* in [4, Chapter 13, §1]).]

LEMMA 4.1. *Let P and Q be posets, $p_1, p_2 \in P$, $q_1, q_2 \in Q$. Assume $p_1 < p_2$ and $q_1 \ngeq q_2$. Then there exists $g \in P^Q$ such that $g(q_1) = p_1$ and $g(q_2) = p_2$.*

Proof. Simply define $g\colon Q \to P$ by

$$g(q) = \begin{cases} p_2 & \text{if } q \geq q_2, \\ p_1 & \text{else} \end{cases}$$

for all $q \in Q$. □

THEOREM 4.2. *Let P and Q be posets such that P is not an antichain and $Q \neq \emptyset$. Then*

$$\nu_{P,Q}: \mathrm{Aut}(P) \times \mathrm{Aut}(Q)^{\partial} \to \mathrm{Aut}(P^Q)$$

is an embedding of ordered groups.

Proof. Note that $\nu := \nu_{P,Q}$ is a group-homomorphism. Let $\lambda, \mu \in \mathrm{Aut}(P)$, $\rho, \sigma \in \mathrm{Aut}(Q)$.

Assume $\lambda \leq \mu$ and $\rho \geq \sigma$. For any $f \in P^Q$,

$$[\nu(\lambda, \rho)](f) = \lambda \circ f \circ \rho^{-1} \leq \mu \circ f \circ \sigma^{-1} = [\nu(\mu, \sigma)](f),$$

so $\nu(\lambda, \rho) \leq \nu(\mu, \sigma)$. Hence ν is order-preserving.

Now assume $\nu(\lambda, \rho) \leq \nu(\mu, \sigma)$. Then, for $p \in P$,

$$\lambda \circ \bar{p} \circ \rho^{-1} \leq \mu \circ \bar{p} \circ \sigma^{-1}.$$

As $Q \neq \emptyset$, we conclude that $\lambda(p) \leq \mu(p)$ for all $p \in P$, so $\lambda \leq \mu$.

Suppose for a contradiction that $\rho \ngeq \sigma$. Then $\rho^{-1} \nleq \sigma^{-1}$, so, for some $q_0 \in Q$, $\rho^{-1}(q_0) \nleq \sigma^{-1}(q_0)$. Let $q_1 := \sigma^{-1}(q_0)$ and $q_2 := \rho^{-1}(q_0)$. As P is not an antichain, there exist $r_1, r_2 \in P$ such that $r_1 < r_2$. Let $p_1 := \mu^{-1}(r_1)$ and $p_2 := \lambda^{-1}(r_2)$. Then $p_1 < p_2$.

By Lemma 1, there exists $g \in P^Q$ such that $g(q_1) = p_1$ and $g(q_2) = p_2$. As

$$(\lambda \circ g \circ \rho^{-1})(q_0) \leq (\mu \circ g \circ \sigma^{-1})(q_0),$$

we see that $(\lambda \circ g)(q_2) \leq (\mu \circ g)(q_1)$, so $\lambda(p_2) \leq \mu(p_1)$, and hence $r_2 \leq r_1$, a contradiction.

Thus, ν is an embedding of ordered groups. □

5. The case of arbitrary exponent

The first definitions are drawn from [22, §1].

A bounded poset P is a *j-poset* if $P = \bigvee \mathcal{J}_c(P)$. A *j-lattice* is a j-poset which is a complete lattice. If both P and P^{∂} are j-posets (j-lattices), then P is a *jm-poset* (*jm-lattice*).

A *bipartite directed graph, or bi-di-graph,* is a triple (X, Y, A) such that $X \cap Y = \emptyset$ and $A \subseteq X \times Y$.

If P is a jm-poset, a *poset of irreducibles of P* is a quintuple (X, Y, A, h, i) where (X, Y, A) is a bi-di-graph and

$$h: X \to \mathscr{J}_c(P) \quad \text{and} \quad i: Y \to \mathscr{M}_c(P)$$

bijections such that $(x, y) \in A$ if and only if $h(x) \nleq i(y)$ $(x \in X, y \in Y)$.

If $R := (X, Y, A, h, i)$ and $R' := (X', Y', A', h', i')$ are posets of irreducibles of a *jm*-poset P, (α, β) is an *isomorphism from* R *to* R' if $\alpha: X \to X'$ and $\beta: Y \to Y'$ are bijections and, for all $x \in X, y \in Y$, $(x, y) \in A$ if and only if $(\alpha(x), \beta(y)) \in A'$.

Note that the "poset of irreducibles" is not in general a poset at all, but merely a bi-di-graph. We shall turn it upside-down and flesh it out so that it truly becomes a poset.

A generalized bi-di-graph is a quintuple $(X, Y, A, \leq_X, \leq_Y)$ such that both $(X, \leq_X)$ and $(Y, \leq_Y)$ are posets and (X, Y, A) is a bi-di-graph.

If P is a *jm*-poset, a *generalized poset of irreducibles of* P is a septuple $(X, Y, A, h, i, \leq_X, \leq_Y)$ such that $(X, Y, A, \leq_X, \leq_Y)$ is a generalized bi-di-graph, (X, Y, A, h, i) is a poset of irreducibles of P, and

$$h: (X, \leq_X) \to \mathscr{J}_c(P) \quad \text{and} \quad i: (Y, \leq_Y) \to \mathscr{M}_c(P)$$

are order-isomorphisms.

If $R := (X, Y, A, h, i, \leq_X, \leq_Y)$ and $R' := (X', Y', A', h', i', \leq_{X'}, \leq_{Y'})$ are generalized posets of irreducibles of a *jm*-poset P, (α, β) is an *isomorphism from* R *to* R' if (α, β) is an isomorphism from (X, Y, A, h, i) to (X', Y', A', h', i') as posets of irreducibles and α and β are order-isomorphisms.

LEMMA 5.1. *Let* P *be a jm-poset,* $R = (X, Y, A, h, i, \leq_X, \leq_Y)$ *a generalized poset of irreducibles of* P. *The relation*

$$\leq_{\bar{R}} := \leq_X \cup \leq_Y \cup \breve{A}$$

on $X \cup Y$ *is a partial order, and we set* $\bar{R} := (X \cup Y, \leq_{\bar{R}})$. *(Here* $\breve{A}$ *is the converse of* A, *i.e.,* $\{(y, x) \in (X \cup Y)^2 \mid (x, y) \in A\}$.*)*

Proof. Note that, for all $u, v \in X \cup Y$, $(u, v) \in \breve{A}$ only if $u \in Y$ and $v \in X$.

The reflexivity of $\leq_{\bar{R}}$ is obvious.

Suppose $u \leq_{\bar{R}} v$ and $v \leq_{\bar{R}} w$, where $u, v, w \in X \cup Y$. If $(u, v) \in \breve{A}$, then $u \in Y$ and $v \in X$, so $(v, w) \in \leq_X$. Thus $h(v) \nleq i(u)$ and $h(v) \leq h(w)$, so $h(w) \nleq i(u)$, i.e., $(u, w) \in \breve{A} \subseteq \leq_{\bar{R}}$. Similarly, if $(v, w) \in \breve{A}$, then $(u, w) \in \leq_{\bar{R}}$. Hence, $\leq_{\bar{R}}$ is transitive.

If $u \leq_{\bar{R}} v$ and $v \leq_{\bar{R}} u$, then $(u, v) \in \leq_X \cup \leq_Y$, so $u = v$. Therefore $\leq_{\bar{R}}$ is antisymmetric. □

LEMMA 5.2. *Let* L *be a jm-lattice,* $R = (X, Y, A, h, i, \leq_X, \leq_Y)$ *a generalized poset of irreducibles of* L. *Then* $\bar{R} = \emptyset$ *if and only if* L *is trivial. The poset* $\bar{R}$ *is* *disconnected if and only if the graph $(X, Y, A \cup \check{A})$ is disconnected. In this case, L is decomposable.*

Proof. The first statement is obvious, as is the necessity of the second statement. Suppose $\bar{R}$ is connected. To prove $(X, Y, A \cup \check{A})$ is connected, it suffices to prove that if $u, v \in \bar{R}$ and $u \leq_X v$ $(u \leq_Y v)$ there exists $w \in Y$ (respectively, X) such that $(u, w), (v, w) \in A$ (respectively, $\check{A}$).

So assume $u, v \in \bar{R}$ and $u \leq_X v$. Then, as $h(u) \neq 0$, there exists $m \in \mathcal{M}_c(L)$ such that $h(u) \nleq m$; hence also $h(v) \nleq m$. If $w = i^{-1}(m)$, then $(u, w), (v, w) \in A$, as desired. We argue dually if $u \leq_Y v$.

The last remark is [22, Theorem 15(a)]. □

LEMMA 5.3. *Let P be a bounded poset and Q any poset. Let $p, r \in P$ and $q, s \in Q$. Then:*

(1) *provided $p \neq 0$, $j(p, q) \leq j(r, s)$ if and only if $p \leq r$ and $q \geq s$;*
(2) *provided $r \neq 1$, $k(p, q) \leq k(r, s)$ if and only if $p \leq r$ and $q \geq s$;*
(3) *$j(p, q) \nleq k(r, s)$ if and only if $p \nleq r$ and $q \leq s$.*

Proof. The first two parts are obvious: see the proof of [19, Lemma 6.5]. For the last part,

$$j(p, q) \leq k(r, s) \quad \Leftrightarrow \quad \text{for all } t \in Q, [j(p, q)](t) \leq [k(r, s)](t)$$
$$\Leftrightarrow \quad q \leq s \text{ implies } p \leq r. \qquad \square$$

PROPOSITION 5.4. *Let P be a jm-poset and Q a poset. Then:*

(1) *P^Q is a jm-poset;*
(2) $\mathcal{J}_c(P^Q) = \{j(p, q) \mid p \in \mathcal{J}_c(P), q \in Q\}$;
(3) $\mathcal{M}_c(P^Q) = \{k(p, q) \mid p \in \mathcal{M}_c(P), q \in Q\}$;
(4) *if $R = (X, Y, A, h, i, \leq_X, \leq_Y)$ and $S = (U, V, B, c, d, \leq_U, \leq_V)$ are generalized posets of irreducibles of P and P^Q, respectively, then the map*

$$\phi: \bar{S} \to \bar{R} \times Q^\partial$$

given by

$$(\phi \circ c^{-1})[j(p_1, q)] = (h^{-1}(p_1), q) \quad \text{and} \quad (\phi \circ d^{-1})[k(p_2, q)] = (i^{-1}(p_2), q)$$

for all $p_1 \in \mathcal{J}_c(L)$, $p_2 \in \mathcal{M}_c(L)$, and $q \in Q$, is an order-isomorphism.

Proof. Without loss of generality, $Q \neq \emptyset$. In this case, (1) follows from [19, Corollary 6.6] and its dual; (2) and (3) follow from [19, Lemma 6.5(iii)] and its dual.

By (2) and (3), ϕ is onto. Let $t_1, t_2 \in U \cup V$. Suppose $t_1, t_2 \in U$, where $c(t_1) = j(p, q)$ and $c(t_2) = j(r, s)$. Then

$$\begin{aligned} t_1 \leq_{\bar{S}} t_2 &\Leftrightarrow j(p, q) \leq j(r, s) \\ &\Leftrightarrow p \leq r \text{ and } q \geq s \\ &\Leftrightarrow h^{-1}(p) \leq h^{-1}(r) \text{ and } q \geq s \\ &\Leftrightarrow \phi(t_1) \leq \phi(t_2). \end{aligned}$$

Similarly, if $t_1, t_2 \in V$, then $t \leq_{\bar{S}} t_2$ if and only if $\phi(t_1) \leq \phi(t_2)$.

If $t_1 \in U$ and $t_2 \in V$, then $t_1 \not\leq_{\bar{S}} t_2$ and $\phi(t_1) \not\leq \phi(t_2)$. Lastly, if $t_1 \in V$ and $t_2 \in U$, where $c(t_2) = j(p, q)$ and $d(t_1) = k(r, s)$, then

$$\begin{aligned} t_1 \leq_{\bar{S}} t_2 &\Leftrightarrow j(p, q) \not\leq k(r, s) \\ &\Leftrightarrow p \not\leq r \text{ and } q \leq s \\ &\Leftrightarrow i^{-1}(r) \leq_{\bar{R}} h^{-1}(p) \text{ and } s \geq q \\ &\Leftrightarrow \phi(t_1) \leq \phi(t_2). \end{aligned}$$

□

The following is established by the argument on pp. 120–121 of [16]:

LEMMA 5.5. *Let X and Y be connected posets, $y \in Y$ and $\alpha \in \mathrm{Aut}(X \times Y)$. Then $\alpha[X \times \{y\}] = A \times B$ for some $A \subseteq X$, $B \subseteq Y$.* □

THEOREM 5.6. *Let L be a jm-lattice that is directly and exponentially indecomposable. Let Q be a non-empty connected poset. Then the natural map*

$$\nu_{L,Q}: \mathrm{Aut}(L) \times \mathrm{Aut}(Q)^{\partial} \to \mathrm{Aut}(L^Q)$$

is an isomorphism of ordered groups.

Proof. By Theorem 4.2, it suffices to prove $\nu_{L,Q}$ is surjective. Let $\gamma \in \mathrm{Aut}(L^Q)$. Let $R = (X, Y, A, h, i, \leq_X, \leq_Y)$ and $S = (U, V, B, c, d, \leq_U, \leq_V)$ be generalized posets of irreducibles of L and L^Q, respectively. Let $\phi: \bar{S} \to \bar{R} \times Q^{\partial}$ be the isomorphism of Proposition 4. The map $(c^{-1} \circ \gamma \circ c, d^{-1} \circ \gamma \circ d)$ is an automorphism of S, and so induces an automorphism $\bar{\gamma}$ of $\bar{S}$; let

$$\Gamma := \phi \circ \bar{\gamma} \circ \phi^{-1} \in \mathrm{Aut}(\bar{R} \times Q^{\partial}).$$

By Lemma 2, $\bar{R}$ is non-empty and connected.

Let $q \in Q$. By Lemma 5, $\Gamma[\bar{R} \times \{q\}] = Z \times W^{\partial}$ for some $Z \subseteq \bar{R}$ and $W \subseteq Q$. By Proposition 3.8, it is clear that $L \cong M^W$ for some complete lattice M, so W has a single element: call it $\rho(q)$. Note that $\rho \in \mathrm{Aut}(Q)$.

Using the fact $\gamma^{-1} \in \mathrm{Aut}(L^Q)$ and symmetry, we see that, for all $q \in Q$, $\Gamma[\bar{R} \times \{q\}] = \mu_q(\bar{R}) \times \{\rho(q)\}$, where $\mu_q \in \mathrm{Aut}(\bar{R})$. If $r \in \bar{R}$, $p, q \in Q$, $p \leq q$, and $\mu_p(r) \neq \mu_q(r)$, then $\mu_p(r) < \mu_q(r)$ and $\rho(p) \leq \rho(q)$. As

$$(\mu_p(r), \rho(p)) < (\mu_q(r), \rho(p)) \leq (\mu_q(r), \rho(q)),$$

we conclude that

$$(r, p) < \Gamma^{-1}(\mu_q(r), \rho(p)) \leq (r, q).$$

Hence $\Gamma^{-1}(\mu_q(r), \rho(p)) = (r, p)$, a contradiction.

As Q is connected, there exists $\mu \in \mathrm{Aut}(\bar{R})$ such that $\mu_q = \mu$ for all $q \in Q$. Moreover, $\mu[X] = X$ and $\mu[Y] = Y$, so by [22, Theorem 6(b)] there exists $\lambda \in \mathrm{Aut}(L)$ such that, for all $p \in L$,

$$\lambda(p) = \bigvee (h \circ \mu \circ h^{-1})[\downarrow_{\mathcal{J}_c(L)} p].$$

Therefore, for all $f \in L^Q$ and $q \in Q$,

$$\begin{aligned}[\gamma(f)](\rho(q)) &= \left[\gamma\left(\bigvee_{p \in Q} j(f(p), p)\right)\right](\rho(q)) \\ &= \left[\bigvee_{p \in Q} \gamma[j(f(p), p)]\right](\rho(q)) \\ &= \bigvee_{p \in Q} [j((\lambda \circ f)(p), \rho(p))](\rho(q)) \\ &= (\lambda \circ f)(q),\end{aligned}$$

so that

$$\gamma(f) = \lambda \circ f \circ \rho^{-1}.$$

□

6. The case of finitely factorable exponent

In this section, unless otherwise specified, we use the following notation, justified by the results of [18, §2]. Let L and Q be such that L is a j-lattice and Q is a connected and directly indecomposable poset. Let $\gamma \in \mathrm{Aut}(L^Q)$. Let $\mathscr{J}_c(L) = \Sigma_{i \in I} A_i$ be a direct sum decomposition into connected components. For each $i \in I$, γ induces

$$\gamma_i : A_i \times Q^\partial \cong B_i \times Q^\partial,$$

where $\mathscr{J}_c(L) = \Sigma_{i \in I} B_i$ is another direct sum decomposition into connected components.

Let

$$I_0(\gamma) := \{i \in I \mid \gamma_i(a_i, q) = (\lambda_i(a_i), \rho_i(q)) \text{ for some } \lambda_i : A_i \cong B_i \text{ and } \rho_i : Q \cong Q \text{ and all } a_i \in A_i, q \in Q\}.$$

Let

$$I_1(\gamma) := I - I_0(\gamma) = \{i \in I \mid \gamma_i(\mu_i(c_i, r), q) = (\nu_i(c_i, q), r) \text{ for some poset } C_i, \mu_i : C_i \times Q^\partial \cong A_i, \nu_i : C_i \times Q^\partial \cong B_i, \text{ and all } c_i \in C_i, q, r \in Q\}.$$

Let

$$E := \nabla\left\{\bigvee_{r \in Q} \mu_i(c_i, r) \mid i \in I_1(\gamma), c_i \in C_i\right\}$$

and

$$F := \nabla\left\{\bigvee_{q \in Q} \nu_i(c_i, q) \mid i \in I_1(\gamma), c_i \in C_i\right\}.$$

For $q \in Q$, let $T_q := \{\rho_i(q) \mid i \in I_0(\gamma)\}$; for $t \in T_q$, let

$$H_{q,t} := \{i \in I_0(\gamma) \mid \rho_i(q) = t\},$$

$$y_{q,t} := \bigvee_{\substack{i \in H_{q,t} \\ a_i \in A_i}} a_i, \quad \text{and} \quad z_{q,t} := \bigvee_{\substack{i \in H_{q,t} \\ a_i \in A_i}} \lambda_i(a_i).$$

Let

$$p_0 := \bigvee_{\substack{i \in I_0(\gamma) \\ a_i \in A_i}} a_i \quad \text{and} \quad p_1 := \bigvee_{\substack{i \in I_1(\gamma) \\ a_i \in A_i}} a_i.$$

Define a function $\iota\colon \mathcal{J}_c(L) \to I$ by $\iota(a) = i$ if and only if $a \in A_i$ (where $a \in \mathcal{J}_c(L)$ and $i \in I$).

LEMMA 6.1. *Let* $a, b \in \mathcal{J}_c(L)$, $p \in L$, $q, r \in Q$, *and* $Y \subseteq \bigcup_{i \in I_0(\gamma)} A_i$ *be such that* $\gamma j(a, q) = j(b, r)$ *and* $a \le p \vee \bigvee Y$. *Then* $a \le p \vee \bigvee\{y \in Y \mid \rho_{\iota(y)}(q) = r\}$.

Proof. We know

$$j(a, q) \le j(p, q) \vee \bigvee_{y \in Y} j(y, q)$$

$$\Rightarrow \quad j(b, r) \le \gamma j(p, q) \vee \bigvee_{y \in Y} j(\lambda_{\iota(y)}(y), \rho_{\iota(y)}(q))$$

$$\Rightarrow \quad j(b, r) \le \gamma j(p, q) \vee \bigvee \{j(\lambda_{\iota(y)}(y), \rho_{\iota(y)}(q)) \mid y \in Y \text{ and } \rho_{\iota(y)}(q) \le r\}$$

$$\Rightarrow \quad j(b, r) \le \gamma j(p, q) \vee \bigvee \{j(\lambda_{\iota(y)}(y), r) \mid y \in Y \text{ such that } \rho_{\iota(y)}(q) \le r\}$$

$$\Rightarrow \quad j(b, r) \le \gamma j(p, q) \vee \bigvee \{j(\lambda_{\iota(y)}(y), r) \mid y \in Y \text{ such that } \rho_{\iota(y)}(q) = r\}$$
$$\vee \bigvee \{j(\lambda_{\iota(y)}(y), r) \mid y \in Y \text{ such that } \rho_{\iota(y)}(q) < r\}$$

$$\Rightarrow \quad j(a, q) \le j(p, q) \vee \bigvee \{j(y, q) \mid y \in Y \text{ such that } \rho_{\iota(y)}(q) = r\}$$
$$\vee \bigvee \{j(y, \rho^{-1}_{\iota(y)}(r)) \mid y \in Y \text{ such that } q < \rho^{-1}_{\iota(y)}(r)\}.$$

Evaluating at q, we see that

$$a \le p \vee \bigvee \{y \in Y \mid \rho_{\iota(y)}(q) = r\}.$$ □

LEMMA 6.2. *Let* $a_m \in A_m$ *for some* $m \in I_0(\gamma)$. *Let* $Y \subseteq \bigcup_{i \in I_0(\gamma)} A_i$. *Then*

$$a_m \le \bigvee Y \vee p_1$$

implies $a_m \le \bigvee Y$.

Proof. For all $q \in Q$,

$$j(a_m, q) \le \bigvee_{y \in Y} j(y, q) \vee \bigvee_{\substack{k \in I_1(\gamma) \\ c_k \in C_k \\ r \in Q}} j(\mu_k(c_k, r), q)$$

Thus, for all $q \in Q$,

$$j(\lambda_m(a_m), \rho_m(q)) \leq \bigvee_{y \in Y} j(\lambda_{\iota(y)}(y), \rho_{\iota(y)}(q)) \vee \bigvee_{\substack{k \in I_1(\gamma) \\ c_k \in C_k \\ r \in Q}} j(v_k(c_k, q), r)$$

$$\Rightarrow \quad \text{for all } q \in Q, \lambda_m(a_m) \leq \bigvee_{y \in Y} \lambda_{\iota(y)}(y) \vee \bigvee_{\substack{k \in I_1(\gamma) \\ c_k \in C_k}} v_k(c_k, q).$$

Let $s, t \in Q$ be such that $s \not\geq t$. Then

$$j(\lambda_m(a_m), \rho_m(s)) \leq \bigvee_{y \in Y} j(\lambda_{\iota(y)}(y), \rho_m(s)) \vee \bigvee_{\substack{k \in I_1(\gamma) \\ c_k \in C_k}} j(v_k(c_k, t), \rho_m(s))$$

implies

$$j(a_m, s) \leq \bigvee_{y \in Y} j(y, \rho_{\iota(y)}^{-1}(\rho_m(s))) \vee \bigvee_{\substack{k \in I_1(\gamma) \\ c_k \in C_k}} j(\mu_k(c_k, \rho_m(s)), t)$$

which implies $a_m \leq \bigvee_{y \in Y} y$. □

LEMMA 6.3. *Let $a_m \in A_m$ for some $m \in I_1(\gamma)$. Let $Z \subseteq \bigcup_{i \in I_1(\gamma)} A_i$. Then*

$$a_m \leq p_0 \vee \bigvee Z$$

implies $a_m \leq \bigvee Z$.

Proof. For some $c_m \in C_m$ and $r \in Q$, $a_m = \mu_m(c_m, r)$. For all $q \in Q$,

$$a_m \leq \bigvee \{a_k \in A_k \mid k \in I_0(\gamma) \text{ and } \rho_k(q) = r\} \vee \bigvee Z$$

by Lemma 1.

Let $s, t \in q$ be such that $s \not\geq t$. Then

$$j(a_m, t) \leq \bigvee \{j(a_k, t) \mid k \in I_0(\gamma), a_k \in A_k, \text{ and } \rho_k(s) = r\} \vee \bigvee_{z \in Z} j(z, t)$$

implies

$$j(v_m(c_m, t), r) \leq \bigvee \{j(\lambda_k(a_k), \rho_k(t)) \mid k \in I_0(\gamma), a_k \in A_k, \text{ and } \rho_k(s) = r\}$$
$$\vee \bigvee_{z \in Z} \gamma j(z, t)$$

If $k \in I_0(\gamma)$, then $s \ngeq t$ implies $\rho_k(s) \ngeq \rho_k(t)$. By evaluating at r, we get

$$j(v_m(c_m, t), r) \leq \bigvee_{z \in Z} \gamma j(z, t)$$

$$\Rightarrow \quad j(a_m, t) \leq \bigvee_{z \in Z} j(z, t)$$

$$\Rightarrow \quad a_m \leq \bigvee Z.$$

□

COROLLARY 6.4. *We have*

$$\downarrow_{\mathcal{J}_c(L)} p_0 = \bigcup_{i \in I_0(\gamma)} A_i \quad \textit{and} \quad \downarrow_{\mathcal{J}_c(L)} p_1 = \bigcup_{i \in I_1(\gamma)} A_i.$$

Proof. The result follows from Lemmas 2 and 3 by setting $Y = Z = \emptyset$. □

PROPOSITION 6.5. *The lattice L is isomorphic to $\downarrow p_0 \times \downarrow p_1$.*

Proof. Let $\Psi: L \to \downarrow p_0 \times \downarrow p_1$ be defined by

$$\Psi(p) = (p \wedge p_0, p \wedge p_1) \qquad \text{for all } p \in L.$$

The map Ψ is clearly order-preserving. If $p, q \in L$ and $p \nleq q$, then there exist $m \in I$ and $a_m \in A_m$ such that $a_m \leq p$ but $a_m \nleq q$. If $m \in I_\varepsilon(\gamma)$ (where ε is 0 to 1), then $a_m \leq p \wedge p_\varepsilon$ but $a_m \nleq q \wedge p_\varepsilon$ so $\Psi(p) \nleq \Psi(q)$. Hence Ψ is an order-embedding.

Let $(u, v) \in \downarrow p_0 \times \downarrow p_1$. By Corollary 4,

$$u \in V \bigcup_{i \in I_0(\gamma)} A_i \quad \text{and} \quad v \in V \bigcup_{i \in I_1(\gamma)} A_i.$$

We claim that $\Psi(u \vee v) = (u, v)$. Clearly $(u, v) \leq \Psi(u \vee v)$. If $\Psi(u \vee v) \nleq (u, v)$, then there exist $m \in I$ and $a_m \in A_m$ such that either $a_m \leq (u \vee v) \wedge p_0$ and $a_m \nleq u$, or $a_m \leq (u \vee v) \wedge p_1$ and $a_m \nleq v$. In either case, $a_m \leq u \vee v$. In the first case, by Corollary 4, $m \in I_0(\gamma)$, so $a_m \leq u$ (Lemma 2). In the second case, by Corollary 4, $m \in I_1(\gamma)$, so $a_m \leq v$ (Lemma 3). This contradiction shows that $\Psi(u, v) \leq (u, v)$, so that $(u, v) = \Psi(u, v)$. Thus Ψ is onto, and so an order-isomorphism. □

The following is immediate.

COROLLARY 6.6. *If L is directly indecomposable, then $I_0(\gamma) = \emptyset$ or $I_1(\gamma) = \emptyset$.* □

The next lemma is also easy.

LEMMA 6.7. *For all $q \in Q$ and $t \in T_q$, $\gamma j(y_{q,t}, q) = j(z_{q,t}, t)$.* □

LEMMA 6.8. *Let $a \in L$, $q \in Q$, and $t \in T_q$. If $a \leq y_{q,t}$, then $\gamma j(a, q) = j(b, t)$ for some $b \in L$.*

Proof. Without loss of generality $a \in A_m$ for some $m \in I$. By Corollary 4, $m \in I_0(\gamma)$. By Lemma 1, $a \leq \bigvee \{a_i \mid i \in H_{q,t}, a_i \in A_i, \text{ and } \rho_i(q) = \rho_m(q)\}$. Thus $\rho_m(q) = t$. □

LEMMA 6.9. *For $q \in Q$, γ restricts to an isomorphism from*

$$\{j(a, q) \mid a \in \downarrow p_0\}$$

onto

$$\nabla \{j(b, t) \mid t \in T_q \text{ and } b \in \downarrow z_{q,t}\}.$$

Proof. Each $a \in \downarrow p_0$ may be represented as

$$\bigvee_{t \in T_q} a_{q,t} \quad \text{where } a_{q,t} \leq y_{q,t} (t \in T_q)$$

by Corollary 4. By Lemma 8 the restriction of γ is an order-embedding.
To show the map is onto, it suffices to consider an element

$$b \in \downarrow_{\mathscr{I}_c(L)} z_{q,t}$$

for some $t \in T_q$. By Corollary 4, $b \in B_m$ for some $m \in I_0(\gamma)$, and

$$\gamma^{-1} j(b, t) = j(a, q)$$

for some $a \in A_m$ (Lemma 8). □

LEMMA 6.10. *For $q \in Q$, let*

$$(b_{q,t})_{t \in T_q}, (b'_{q,t})_{t \in T_q} \in \prod_{t \in T_q} \downarrow z_{q,t}.$$

Then

$$\bigvee_{t \in T_q} b_{q,t} \leq \bigvee_{t \in T_q} b'_{q,t} \quad \text{if and only if} \quad b_{q,t} \leq b'_{q,t} \quad \text{for all } t \in T_q.$$

Proof. Sufficiency is obvious. For necessity, fix $t \in T_q$. Without loss of generality, $b_{q,t} \in \mathscr{J}_c(L)$. For each $u \in T_q$, there is a subset $Z_{q,u} \subseteq \mathscr{J}_c(L)$ such that $b'_{q,u} = \bigvee Z_{q,u}$. By Corollary 4,

$$Z_{q,u} \subseteq \bigcup_{i \in I_0(\gamma)} B_i$$

for all $u \in T_q$. By Lemma 9, for all $u \in T_q$ and $z \in Z_{q,u}$, $\gamma^{-1}j(z, u) = j(y, q)$ for some $y \in \mathscr{J}_c(L)$. Lemma 1 for γ^{-1} implies that $b_{q,t} \leq \bigvee Z_{q,t} = b'_{q,t}$. □

LEMMA 6.11. *Let $q \in Q$. For $f \in \nabla\{j(b, t) \mid t \in T_q$ and $b \in \downarrow z_{q,t}\}$, there is a unique family*

$$\Psi(f) := (b_{q,t})_{t \in T_q} \in \prod_{t \in T_q} \downarrow z_{q,t}$$

such that

$$f = \bigvee_{t \in T_q} j(b_{q,t}, t).$$

The map

$$\Psi : \nabla\{j(b, t) \mid t \in T_q \text{ and } b \in \downarrow z_{q,t}\} \to \prod_{t \in T_q} \downarrow z_{q,t}$$

is an order-isomorphism.

Proof. The function f equals

$$\bigvee_{t \in T_q} j(b_{q,t}, t)$$

for some family

$$(b_{q,t})_{t \in T_q} \in \prod_{t \in T_q} \downarrow z_{q,t}.$$

If

$$(b'_{q,t})_{t \in T_q} \in \prod_{t \in T_q} \downarrow z_{q,t}$$

is another such family, then by Lemma 10 $b_{q,t} = b'_{q,t}$ for all $t \in T_q$. Hence Ψ is well-defined. By Lemma 10, it is an order-embedding. It is clearly onto, hence an order-isomorphism. □

The next corollary follows from Lemmas 9 and 11.

COROLLARY 6.12. *For $q \in Q$, there is an isomorphism from $\downarrow p_0$ onto*

$$\prod_{t \in T_q} \downarrow z_{q,t}. \qquad \square$$

COROLLARY 6.13. *Suppose L is directly indecomposable and $I_1(\gamma) = \emptyset$. Then there is a map $\rho \in \mathrm{Aut}(Q)$ such that $\rho = \rho_i$ for all $i \in I$.*

Proof. By Corollary 12 and the fact $I_1(\gamma) = \emptyset$, L is isomorphic to

$$\prod_{t \in T_q} \downarrow z_{q,t}$$

for each $q \in Q$. Hence, for each $q \in Q$, T_q is a singleton. Let $\rho(q)$ denote the sole member of T_q. For all $i \in I$ and $q \in Q$, $\rho_i(q) \in T_q = \{\rho(q)\}$. $\square$

PROPOSITION 6.14. *If L is directly indecomposable and $I_1(\gamma) = \emptyset$, then there exist $\lambda \in \mathrm{Aut}(L)$ and $\rho \in \mathrm{Aut}(Q)$ such that $\gamma = v_{L,Q}(\lambda, \rho)$. Indeed, λ may be defined by the formula $\overline{\lambda(p)} = \gamma(\bar{p})$ for all $p \in L$.*

Proof. For each $i \in I$ and $p \in L$, let

$$\mathscr{J}_i(p) := \downarrow_{A_i} p.$$

Let ρ be the map of Corollary 13. For all $f \in L^Q$

$$\begin{aligned}
\gamma(f) &= \gamma\left(\bigvee_{q \in Q} j(f(q), q)\right) \\
&= \bigvee_{q \in Q} \gamma[j(f(q), q)] \\
&= \bigvee_{q \in Q} \bigvee_{i \in I} \gamma\left[j\left(\bigvee_{a_i \in \mathscr{J}_i[f(q)]} a_i, q\right)\right] \\
&= \bigvee_{q \in Q} \bigvee_{i \in I} \bigvee_{a_i \in \mathscr{J}_i[f(q)]} j(\lambda_i(a_i), \rho(q)) \\
&= \bigvee_{q \in Q} j\left(\bigvee_{i \in I} \bigvee_{a_i \in \mathscr{J}_i[f(q)]} \lambda_i(a_i), \rho(q)\right).
\end{aligned}$$

In particular, for all $p \in L$

$$\gamma(\bar{p}) = \overline{\bigvee_{i \in I} \bigvee_{a_i \in \mathscr{I}_i(p)} \lambda_i(a_i)}.$$

Hence the map λ is well-defined. Clearly it is an isomorphism.

For all $f \in L^Q$ and $q \in Q$,

$$\begin{aligned}[\gamma(f)](\rho(q)) = \lambda(f(q)) \quad &\Rightarrow \quad [\gamma(f)](q) = \lambda[f(\rho^{-1}(q))] \\ &\Rightarrow \quad \gamma(f) = \lambda \circ f \circ \rho^{-1}. \end{aligned}$$

□

The following result is essentially a special case of [18, Theorem 11].

PROPOSITION 6.15. *Suppose* $I_0(\gamma) = \emptyset$. *Then*

$$E = \{p \in L \mid \gamma(\bar{p}) \textit{ is a constant}\} \textit{ and } F = \{p \in L \mid \gamma^{-1}(\bar{p}) \textit{ is a constant}\}.$$

The map $\mathring{\gamma}: E \to F$ *defined by* $\overline{\mathring{\gamma}(e)} = \gamma(\bar{e})$ *for all* $e \in E$ *is an isomorphism. The maps* $\lambda: L \to F^Q$ *and* $\mu: L \to E^Q$ *defined by*

$$\lambda(p) = \gamma(\bar{p}) \quad \textit{and} \quad \mu(p) = \gamma^{-1}(\bar{p}) \qquad \textit{for all } p \in L$$

are isomorphisms.

The map $\tilde{\gamma}: (F^Q)^Q \to (E^Q)^Q$ *defined by*

$$\tilde{\gamma}(h)(r)(q) = \mathring{\gamma}^{-1}[h(q)(r)]$$

for all $h \in (F^Q)^Q$ *and* $q, r \in Q$ *is an isomorphism, and for all* $f \in L^Q$

$$\tilde{\gamma}(\lambda \circ f) = \mu \circ [\gamma(f)].$$

Proof. For $i \in I$, $c_i \in C_i$, and $r \in Q$,

$$\begin{aligned}\gamma(\overline{\mu_i(c_i, r)}) &= \gamma\left[\bigvee_{q \in Q} j(\mu_i(c_i, r), q)\right] \\ &= \bigvee_{q \in Q} j(v_i(c_i, q), r) \in F^Q\end{aligned}$$

so λ is well-defined. Working backwards we see that λ is onto, so is an isomorphism. Moreover,

$$\gamma\left(\overline{\bigvee_{r\in Q}\mu_i(c_i,r)}\right)=\overline{\bigvee_{\substack{q\in Q\\ r\in Q}} j(\nu_i(c_i,q),r)}$$
$$=\overline{\bigvee_{q\in Q}\nu_i(c_i,q)}.$$

Hence, for all $e\in E$, $\gamma(\bar e)$ is constant with image in F, so $\mathring{\gamma}$ is well-defined. It is clearly an order-isomorphism with inverse $(\mathring{\gamma})^{-1}=(\gamma^{-1})^{\circ}$.

If $p\in L$ and $\gamma(\bar p)=\overline{p'}$ for some $p'\in L$, then p' is a join of elements of the form $\nu_i(c_i,q)$ $(i\in I, c_i\in C_i, q\in Q)$. Hence $\overline{p'}$ is a join of functions of the form $\bigvee_{r\in Q} j(\nu_i(c_i,q),r)$ and $\gamma^{-1}(\overline{p'})=\bar p$ is a join of functions of the form $j(\bigvee_{r\in Q}\mu_i(c_i,r),q)$. Therefore p lies in E.

The map $\tilde\gamma$ is well-defined and an isomorphism with inverse $\widetilde{\gamma^{-1}}$. Let $i\in I$, $c_i\in C_i$, $q,r,s,t\in Q$. Then

$$\begin{aligned}\mu[\gamma(j(\mu_i(c_i,r),q))(t)](s)&=\mu[j(\nu_i(c_i,q),r)(t)](s)\\&=\gamma^{-1}[\overline{j(\nu_i(c_i,q),r)(t)}](s)\\&=\begin{cases}\gamma^{-1}(\overline{\nu_i(c_i,q)})(s) & \text{if } r\le t,\\ 0 & \text{else}\end{cases}\\&=\begin{cases}\bigvee_{v\in Q} j(\mu_i(c_i,v),q)(s) & \text{if } r\le t,\\ 0 & \text{else}\end{cases}\\&=\begin{cases}\bigvee_{v\in Q}\mu_i(c_i,v) & \text{if } q\le s \text{ and } r\le t,\\ 0 & \text{else.}\end{cases}\end{aligned}$$

On the other hand,

$$\begin{aligned}\tilde\gamma(\lambda\circ j(\mu_i(c_i,r),q))(t)(s)&=\mathring{\gamma}^{-1}[(\lambda\circ j(\mu_i(c_i,r),q))(s)(t)]\\&=\mathring{\gamma}^{-1}[\gamma[\overline{j(\mu_i(c_i,r),q)(s)}](t)]\\&=\begin{cases}\mathring{\gamma}^{-1}[\gamma(\overline{\mu_i(c_i,r)})(t)] & \text{if } q\le s,\\ 0 & \text{else}\end{cases}\\&=\begin{cases}\mathring{\gamma}^{-1}\left(\overline{\bigvee_{u\in Q}\nu_i(c_i,u)}\right) & \text{if } q\le s \text{ and } r\le t,\\ 0 & \text{else}\end{cases}\\&=\begin{cases}\bigvee_{v\in Q}\mu_i(c_i,v) & \text{if } q\le s \text{ and } r\le t,\\ 0 & \text{else.}\end{cases}\end{aligned}$$

Thus, for all $f \in L^Q$,

$$\tilde{\gamma}(\lambda \circ f) = \mu \circ [\gamma(f)]. \qquad \square$$

The following proposition may now be proven along the lines of [18, Theorem 12], using Corollary 6, Proposition 14, and Proposition 15 in place of the theorems cited in its proof (which stipulate as one of their hypotheses that the exponent satisfy the ascending chain condition).

PROPOSITION 6.16. *Let L be a directly indecomposable j-lattice and Q a directly indecomposable connected poset such that L is exponentially indecomposable with respect to* $\{Q\}$. *Then for all* $n \geq 0$, v_{L,Q^n} *is an isomorphism.* □

Similarly, the next proposition may be proven along the same lines as [18, theorem 14], using Proposition 16 in place of [18, Theorem 12] (cited in its proof). As the base is not an antichain and the exponent is non-empty, the conclusion of [18, Lemma 13], used in the proof, applies (see §4). (Note that our result for j-lattices is slightly sharper than the one for j-posets in [18, Theorem 14]; in that theorem, the base is merely assumed to be exponentially indecomposable.)

PROPOSITION 6.17. *Let L be a j-lattice. Let Q be a finitely factorable connected poset. Let* $\mathcal{Q}$ *be the class of direct factors of Q. Assume L is both directly indecomposable and exponentially indecomposable with respect to* $\mathcal{Q}$. *Then the natural map*

$$v_{L,Q}: \mathrm{Aut}(L) \times \mathrm{Aut}(Q)^{\partial} \to \mathrm{Aut}(L^Q)$$

is an isomorphism of ordered groups. □

Hence we get the theorem below.

THEOREM 6.18. *Let L be a j-lattice which is directly and exponentially indecomposable. Let Q be a finitely factorable connected poset. Then the natural map*

$$v_{L,Q}: \mathrm{Aut}(L) \times \mathrm{Aut}(Q)^{\partial} \to \mathrm{Aut}(L^Q)$$

is an isomorphism of ordered groups. □

7. The case of arbitrary base

The following definitions appear in [9], [15, §1], and [20, §I]. Recall that a non-empty subset D of a poset is *directed* if every finite subset of D has an upper bound in D.

We shall call a poset P a *pre-CPO* if every directed subset D has a supremum, denoted $\bigsqcup D$. (The special notation, which is standard, serves as a convenient reminder that the set under consideration is directed.) If it also has a 0 it is called a *CPO* (short for *chain-complete partially ordered set*).

Recall that an element k of a poset P is *compact* if, for all directed $D \subseteq P$ for which $\bigsqcup D$ exists and $k \leq \bigsqcup D$, there is an element $d \in D$ such that $k \leq d$. The set of all compact elements of P is denoted $\kappa(P)$.

An *algebraic poset* is a pre-CPO such that every element is a join of a directed set of compact elements. An *algebraic lattice* is an algebraic poset that is also a complete lattice.

An *ideal* of a poset P is a directed down-set. The family of all ideals of P is denoted P^σ. If $\sigma_P: P \to P^\sigma$ is the canonical map $\sigma_P(p) = \downarrow p$ $(p \in P)$, then (P^σ, σ_P) is called the *ideal completion of P*.

The first statement of the next proposition is contained in [15, §2] and the final two statements are straightforward.

PROPOSITION 7.1. *The maps* $P \mapsto P^\sigma$ *(P a poset) and* $A \mapsto \kappa(A)$ *(A an algebraic poset) are functors yielding an equivalence between the categories of posets with order-preserving maps and algebraic posets with morphisms that preserve directed joins and compact elements. Indeed,* $\kappa[P^\sigma] = \operatorname{Im} \sigma_P$ *and the map* $a \mapsto \downarrow_{\kappa(A)} a$ $(a \in A)$ *is an isomorphism from A to* $[\kappa(A)]^\sigma$.

Moreover, if $\gamma: P \cong Q$ *(P, Q posets) then* $\gamma^\sigma: P^\sigma \cong Q^\sigma$ *is the unique isomorphism such that*

$$\gamma^\sigma \circ \sigma_P = \sigma_Q \circ \gamma.$$

Hence, for $\gamma, \delta \in \operatorname{Aut}(P)$, $\gamma^\sigma = \delta^\sigma$ *implies* $\gamma = \delta$. □

COROLLARY 7.2. *A poset P is directly indecomposable if and only if* P^σ *is.*

Proof. Without loss of generality, $P \neq \emptyset$. First, note that P is trivial if and only if P^σ is. Second, by [15, Corollary 4(1)], the class of algebraic posets is closed under finite direct products, so $P \cong Q \times R$ implies $P^\sigma \cong Q^\sigma \times R^\sigma$. Third, if $\phi: P^\sigma \cong Q \times R$, where Q and R are posets, then Q and R are algebraic posets and

$$\phi[\kappa(P^\sigma)] = \kappa(Q) \times \kappa(R).$$ □

The following result is proven for lattices in [12, Theorem 3.1], but holds more generally for $\vee$-semilattices. (See the remark in [19, §5].)

PROPOSITION 7.3. *Let L be a $\vee$-semilattice and Q a finite poset. Then there is a unique isomorphism $\Psi: (L^Q)^\sigma \to (L^\sigma)^Q$ such that, for all $f \in L^Q$,*

$$(\Psi \circ \sigma_{L^Q})(f) = \sigma_L \circ f. \qquad \square$$

COROLLARY 7.4. *A lattice L is exponentially indecomposable with respect to $\mathscr{P}_{\text{fin}}$ if and only if L^σ is.*

Proof. Without loss of generality, L is non-trivial. Suppose $L \cong P^Q$ where $P \in \mathscr{P}$ and $Q \in \mathscr{P}_{\text{fin}}$. As L is non-trivial, $Q \neq \emptyset$, so P is a lattice. By Proposition 3, $L^\sigma \cong (P^\sigma)^Q$.

Conversely if $L^\sigma \cong A^Q$ where $A \in \mathscr{P}$ and $Q \in \mathscr{P}_{\text{fin}}$, then, again, $Q \neq \emptyset$ and A is a lattice. Clearly A is a pre-CPO.

Now we prove that $f \in \kappa(A^Q)$ implies $f \in [\kappa(A)]^Q$.

Fix $q_0 \in Q$. Define, for each $a \in A$, the function $f_a: Q \to A$ by

$$f_a(q) = \begin{cases} a & \text{if } q \leq q_0, \\ f(q) \vee a & \text{else.} \end{cases}$$

Then $f_a \in A^Q$.

If $D \subseteq A$ is directed and $f(q_0) \leq \bigsqcup D$, then $\{f_d \mid d \in D\}$ is directed and $f \leq \bigsqcup\{f_d \mid d \in D\}$. Hence $f \leq f_d$ for some $d \in D$, so $f(q_0) \leq d$. Therefore, $f(q_0) \in \kappa(A)$.

The above and the fact L^σ is an algebraic poset imply that A is an algebraic poset. Further, $\kappa(A)$ is a $\vee$-semilattice. By Propositions 1 and 3,

$$L \cong \kappa(L^\sigma) \cong \kappa(A^Q) \cong [\kappa(A)]^Q. \qquad \square$$

PROPOSITION 7.5. *Let L be a lattice. Then $L^{\sigma\partial\sigma}$ is an algebraic lattice.*

Proof. It is easy to see that, if M is a lattice, the join of every non-empty subset of M^σ exists. If, in addition, M has a 0, then M^σ is a complete lattice. As $L^{\sigma\partial}$ is a lattice with 0, $L^{\sigma\partial\sigma}$ is a complete lattice. $\square$

COROLLARY 7.6. *Let L be a directly and exponentially indecomposable lattice. Then*:

(1) *$L^{\sigma\partial\sigma\partial}$ is directly indecomposable,*

(2) *$L^{\sigma\partial\sigma\partial}$ is exponentially indecomposable with respect to $\mathscr{P}_{\text{fin}}$, and*

(3) *$L^{\sigma\partial\sigma\partial}$ is a j-lattice.*

Proof. By Proposition 5, $L^{\sigma\partial\sigma\partial}$ is the dual of an algebraic lattice. It follows from [4, Theorem 8.8.16] that $L^{\sigma\partial\sigma\partial} = \nabla\mathscr{J}_c(L^{\sigma\partial\sigma\partial})$. The rest of the corollary follows from Corollaries 2 and 4. □

THEOREM 7.7. *Let L be a directly and exponentially indecomposable lattice. Let Q be a finite non-empty connected poset. Then the natural map*

$$\nu_{L,Q}\colon \mathrm{Aut}(L) \times \mathrm{Aut}(Q)^{\partial} \to \mathrm{Aut}(L^Q)$$

is an isomorphism of ordered groups.

Proof. By Theorem 4.2, it suffices to prove that $\nu_{L,Q}$ is onto. For lattices X and Y, let $\partial_X\colon X \to X^{\partial}$ be the dual-isomorphism $\partial_X(x) = x$ $(x \in X)$. If $\beta\colon X \cong Y$, let $\beta^{\partial}\colon X^{\partial} \to Y^{\partial}$ be the unique isomorphism such that

$$\beta^{\partial} \circ \partial_X = \partial_Y \circ \beta.$$

Let X^{τ}, τ_X, $\tau\tau_X$, and β^{τ} denote $X^{\sigma\partial}$, $\partial_{X^{\sigma}} \circ \sigma_X$, $\tau_{X^{\tau}} \circ \tau_X$, and $\beta^{\sigma\partial}$ respectively. Thus $\beta^{\tau}\colon X^{\tau} \to Y^{\tau}$ is the unique isomorphism such that

$$\beta^{\tau} \circ \tau_X = \tau_Y \circ \beta,$$

and $\beta^{\tau\tau}\colon X^{\tau\tau} \to Y^{\tau\tau}$ is the unique isomorphism such that

$$\beta^{\tau\tau} \circ \tau\tau_X = \tau\tau_Y \circ \beta.$$

By Proposition 3, there is a unique isomorphism

$$\Psi_1\colon (L^Q)^{\tau} \to (L^{\tau})^{Q^{\partial}}$$

such that, for all $f \in L^Q$, $(\Psi_1 \circ \tau_{L^Q})(f) = \tau_L \circ f \circ \partial_Q^{-1}$. Hence there is an isomorphism

$$\Psi_1^{\tau}\colon (L^Q)^{\tau\tau} \to (L^{\tau})^{Q^{\partial\tau}}$$

such that, for all $f \in L^Q$,

$$\begin{aligned}\Psi_1^{\tau}(\tau\tau_{L^Q}(f)) &= (\tau_{(L^{\tau})^{Q^{\partial}}} \circ \Psi_1 \circ \tau_{L^Q})(f)\\ &= \tau_{(L^{\tau})^{Q^{\partial}}}(\tau_L \circ f \circ \partial_Q^{-1}).\end{aligned}$$

Similarly, there is a unique isomorphism

$$\Psi_2: (L^\tau)^{Q^{\partial\tau}} \to (L^{\tau\tau})^Q$$

such that, for all $g \in (L^\tau)^{Q^\partial}$,

$$(\Psi_2 \circ \tau_{(L^\tau)^{Q^\partial}})(g) = \tau_{L^\tau} \circ g \circ \partial_Q.$$

Let $\Psi := \Psi_2 \circ \Psi_1^\tau: (L^Q)^{\tau\tau} \cong (L^{\tau\tau})^Q$. Then, for all $f \in L^Q$,

$$(\Psi \circ \tau\tau_{L^Q})(f) = \tau\tau_L \circ f.$$

Let $\gamma \in \mathrm{Aut}(L^Q)$. The map $\Psi \circ \gamma^{\tau\tau} \circ \Psi^{-1} \in \mathrm{Aut}((L^{\tau\tau})^Q)$.

By Corollary 6 and Proposition 6.17, there exist $\mu \in \mathrm{Aut}(L^{\tau\tau})$ and $\rho \in \mathrm{Aut}(Q)$ such that

$$\Psi \circ \gamma^{\tau\tau} \circ \Psi^{-1} = \nu_{L^{\tau\tau},Q}(\mu, \rho).$$

Clearly there exists $\lambda \in \mathrm{Aut}(L)$ such that $\mu = \lambda^{\tau\tau}$.

We claim that

$$\Psi \circ (\nu_{L,Q}(\lambda, \rho))^{\tau\tau} \circ \Psi^{-1} = \nu_{L^{\tau\tau},Q}(\lambda^{\tau\tau}, \rho).$$

For all $f \in L^Q$,

$$\begin{aligned}
[\Psi^{-1} \circ \nu_{L^{\tau\tau},Q}(\lambda^{\tau\tau}, \rho) \circ \Psi](\tau\tau_{L^Q}(f)) &= [\Psi^{-1} \circ \nu_{L^{\tau\tau},Q}(\lambda^{\tau\tau}, \rho)](\tau\tau_L \circ f) \\
&= \Psi^{-1}(\lambda^{\tau\tau} \circ \tau\tau_L \circ f \circ \rho^{-1}) \\
&= \Psi^{-1}(\tau\tau_L \circ \lambda \circ f \circ \rho^{-1}) \\
&= \Psi^{-1}(\tau\tau_L \circ [\nu_{L,Q}(\lambda, \rho)](f)) \\
&= \Psi^{-1}((\Psi \circ \tau\tau_{L^Q})([\nu_{L,Q}(\lambda, \rho)](f))) \\
&= \tau\tau_{L^Q}([\nu_{L,Q}(\lambda, \rho)](f)).
\end{aligned}$$

Therefore

$$\Psi^{-1} \circ \nu_{L^{\tau\tau},Q}(\lambda^{\tau\tau}, \rho) \circ \Psi \circ \tau\tau_{L^Q} = \tau\tau_{L^Q} \circ \nu_{L,Q}(\lambda, \rho)$$

so

$$\Psi^{-1} \circ v_{L^{\tau\tau},Q}(\lambda^{\tau\tau}, \rho) \circ \Psi = (v_{L,Q}(\lambda, \rho))^{\tau\tau}.$$

Hence $\gamma^{\tau\tau} = v_{L,Q}(\lambda, \rho)^{\tau\tau}$, so $\gamma = v_{L,Q}(\lambda, \rho)$ by Proposition 1. □

8. The Dedekind–MacNeille completion of a function space

Let P be a poset, $Q \subseteq P$. The set of upper bounds of $Q \in P$ is denoted Q^u; the set of lower bounds of Q in P is denoted Q^l. The set

$$\mathbf{DM}(P) := \{Q \subseteq P \mid Q = Q^{ul}\}$$

partially ordered by inclusion is the *Dedekind–MacNeille completion of P*. The map

$$\varphi_P : P \to \mathbf{DM}(P)$$

defined by $\varphi_P(p) = \downarrow p$ $(p \in P)$ is the *canonical embedding of P into* $\mathbf{DM}(P)$. We often denote Im φ_P by $\bar{P}$. (Cf. [9, 2.31], [21, Definition 11.1].)

LEMMA 8.1. *Let P be a bounded poset, Q any poset, L a complete lattice,* $\varphi \in L^P$ *such that* Im φ *is join- and meet-dense in L,* $\varphi(0_P) = 0_L$, *and* $\varphi(1_P) = 1_L$. *Then*

$$\{\varphi \circ j(p, q) \mid p \in P, q \in Q\}$$

is join-dense in L^Q *and*

$$\{\varphi \circ k(p, q) \mid p \in P, q \in Q\}$$

is meet-dense in L^Q.

Proof. It suffices, by duality, to prove the first assertion. As Im φ is join-dense in L,

$$f(q) = \bigvee_L (\downarrow_{\mathrm{Im}\,\varphi} f(q))$$

for all $f \in L^Q$, $q \in Q$.

For all $f \in L^Q$, let

$$\mathscr{F}(f) := \{\varphi \circ j(p, q) \mid p \in P, q \in Q, \varphi(p) \leq f(q)\}.$$

First we show that $f \in \mathscr{F}(f)^u$.

Assume $p \in P$, $q, r \in Q$, and $\varphi(p) \leq f(q)$. It suffices to prove that

$$[\varphi \circ j(p, q)](r) \leq f(r).$$

If $q \leq r$, then

$$[\varphi \circ j(p, q)](r) = \varphi(p) \leq f(q) \leq f(r).$$

If $q \nleq r$, then

$$[\varphi \circ j(p, q)](r) = \varphi(0_P) = 0_L \leq f(r).$$

Next we show that $f = \bigvee \mathscr{F}(f)$. Let $g \in \mathscr{F}(f)^u$. For all $p \in P, q \in Q$ such that $\varphi(p) \leq f(q)$,

$$[\varphi \circ j(p, q)](q) \leq g(q),$$

so $\varphi(p) \leq g(q)$. Hence for all $q \in Q$,

$$g(q) \in \{\varphi(p) \mid p \in P, \varphi(p) \leq f(q)\}^u = (\downarrow_{\mathrm{Im}\,\varphi} f(q))^u$$

so $g(q) \geq f(q)$. Therefore $g \geq f$.

We have shown that every $f \in L^Q$ equals $\bigvee \mathscr{F}(f)$, so

$$\{\varphi \circ j(p, q) \mid p \in P, q \in Q\}$$

is join-dense in L^Q. □

COROLLARY 8.2. *Let P be a bounded poset and Q any poset. Then*

$$\{\varphi_P \circ j(p, q), \varphi_P \circ k(p, q) \mid p \in P, q \in Q\}$$

is join- and meet-dense in $\mathbf{DM}(P)^Q$.

Proof. By [9, Theorem 2.36(i)], $\bar{P}$ is join- and meet-dense in $\mathbf{DM}(P)$. By [9, Theorem 2.33(ii)], $\varphi_P(0_P) = 0_{\mathbf{DM}(P)}$ and $\varphi_P(1_P) = 1_{\mathbf{DM}(P)}$. □

Let W and X be posets, $A \subseteq W$, $B \subseteq X$. Let

$$\mathrm{Iso}(W, X) := \{f\colon W \to X \mid f \text{ is an isomorphism}\},$$

$$\mathrm{Iso}_{A,B}(W, X) := \{f \in \mathrm{Iso}(W, X) \mid f[A] = B\},$$

and

$$\mathrm{Aut}_A(W) := \mathrm{Iso}_{A,A}(W, W).$$

The next lemma is easy.

LEMMA 8.3. *Let W, X, Y, and Z be posets and $\alpha\colon W \to Y$, $\beta\colon X \to Z$ isomorphisms. Let $A \subseteq W$, $B \subseteq X$. Define $\Phi\colon X^W \to Z^Y$ by*

$$\Phi(f) = \beta \circ f \circ \alpha^{-1}$$

for all $f \in X^W$. Then Φ is an order-isomorphism and

$$\Phi[\mathrm{Iso}_{A,B}(W, X)] = \mathrm{Iso}_{\alpha[A],\beta[B]}(Y, Z).$$

In particular, the map $\Phi'\colon \mathrm{Aut}_A(W) \to \mathrm{Aut}_{\alpha[A]}(Y)$ defined by

$$\Phi'(f) = \alpha \circ f \circ \alpha^{-1} \quad (f \in \mathrm{Aut}_A(W))$$

is an order-isomorphism. □

PROPOSITION 8.4. *Let U and V be posets. There is an order-isomorphism*

$$\Omega\colon \mathrm{Iso}(U, V) \to \mathrm{Iso}_{\bar{U},\bar{V}}(\mathbf{DM}(U), \mathbf{DM}(V))$$

with the following property: for each $f \in \mathrm{Iso}(U, V)$, $\Omega(f)\colon \mathbf{DM}(U) \to \mathbf{DM}(V)$ is the unique order-isomorphism such that

$$\Omega(f) \circ \varphi_U = \varphi_V \circ f.$$

Proof. Note that if $f\colon U \cong V$ and $T \subseteq U$, then $f[T^{ul}] = f[T]^{ul}$. For $f \in \mathrm{Iso}(U, V)$, define $\Omega(f)\colon \mathbf{DM}(U) \to \mathbf{DM}(V)$ by

$$(\Omega(f))(T) = f[T]$$

for all $T \in \mathbf{DM}(U)$. The above shows $\Omega(f)$ is well-defined for all $f \in \text{Iso}(U, V)$ and clearly

$$\Omega(f) \circ \varphi_U = \varphi_V \circ f$$

and

$$\Omega(f) \in \text{Iso}_{\bar{U},\bar{V}}(\mathbf{DM}(U), \mathbf{DM}(V)).$$

Let $f, g \in \text{Iso}(U, V)$. As each element of $\mathbf{DM}(V)$ is a down-set, $f \leq g$ implies $f[T] \subseteq g[T]$ for all $T \in \mathbf{DM}(U)$, so $\Omega(f) \leq \Omega(g)$. Conversely, $\Omega(f) \leq \Omega(g)$ implies that

$$\downarrow f(u) = (\Omega(f))(\downarrow u) \leq (\Omega(g))(\downarrow u) = \downarrow g(u)$$

for all $u \in U$. Hence $f \leq g$. Thus, Ω is an order-embedding.

Let $h \in \text{Iso}_{\bar{U},\bar{V}}(\mathbf{DM}(U), \mathbf{DM}(V))$. Let $f\colon U \to V$ be the map such that $h \circ \varphi_U = \varphi_V \circ f$, obviously an isomorphism. As $\bar{U}$ is join-dense in $\mathbf{DM}(U)$, $\Omega(f) = h$, so Ω is an order-isomorphism. □

COROLLARY 8.5. *Let L be a complete lattice, U, V posets such that V is join- and meet-dense in L, and $f\colon U \cong V$. Then there is an order-isomorphism $\Gamma\colon \mathbf{DM}(U) \to L$ such that*

$$\iota_V \circ f = \Gamma \circ \varphi_U.$$

Proof. By [9, Theorem 2.36(iii)], there exists an isomorphism $\chi\colon L \cong \mathbf{DM}(V)$ such that

$$\chi \restriction_V = \varphi_V.$$

By Proposition 4, there exists an isomorphism $\Omega(f)\colon \mathbf{DM}(U) \cong \mathbf{DM}(V)$ such that

$$\Omega(f) \circ \varphi_U = \varphi_V \circ f.$$

Hence $\Omega(f) \circ \varphi_U = \chi \circ f$. Let $\Gamma := \chi^{-1} \circ \Omega(f)$. □

THEOREM 8.6. *Let P be a bounded poset and Q any poset. There is a unique isomorphism $\Gamma\colon \mathbf{DM}(P^Q) \to \mathbf{DM}(P)^Q$ such that, for all $f \in P^Q$,*

$$(\Gamma \circ \varphi_{P^Q})(f) = \varphi_P \circ f.$$

Proof. The map $\Gamma_1 : P^Q \to \mathbf{DM}(P)^Q$ defined by $\Gamma_1(f) = \varphi_P \circ f$ for all $f \in P^Q$ is an order-embedding. By Corollary 2, $\operatorname{Im} \Gamma_1 = \bar{P}^Q$ is join- and meet-dense in $\mathbf{DM}(P)^Q$. By Corollary 5, there is an order-isomorphism

$$\Gamma : \mathbf{DM}(P^Q) \to \mathbf{DM}(P)^Q$$

such that $\Gamma_1 = \Gamma \circ \varphi_{PQ}$. □

PROPOSITION 8.7. *Let P be a bounded poset, Q any poset, and $\gamma \in \operatorname{Aut}(P^Q)$. Then there exists a unique $\hat{\gamma} \in \operatorname{Aut}_{\bar{P}Q}(\mathbf{DM}(P)^Q)$ such that, for all $f \in P^Q$,*

$$\hat{\gamma}(\varphi_P \circ f) = \varphi_P \circ \gamma(f).$$

We call $\hat{\gamma}$ the automorphism of $\mathbf{DM}(P)^Q$ induced by γ.

Proof. By Proposition 4, there exists $\Omega(\gamma) \in \operatorname{Aut}_{\overline{PQ}}(\mathbf{DM}(P^Q))$ such that

$$\Omega(\gamma) \circ \varphi_{PQ} = \varphi_{PQ} \circ \gamma.$$

By Theorem 6, there is an isomorphism $\Gamma : \mathbf{DM}(P^Q) \to \mathbf{DM}(P)^Q$ such that

$$(\Gamma \circ \varphi_{PQ})(f) = \varphi_P \circ f$$

for all $f \in P^Q$. Thus $\Gamma[\overline{P^Q}] = \bar{P}^Q$. By Lemma 3, there exists

$$\hat{\gamma} \in \operatorname{Aut}_{\bar{P}Q}(\mathbf{DM}(P)^Q)$$

such that $\hat{\gamma} \circ \Gamma = \Gamma \circ \Omega(\gamma)$. Hence, for all $f \in P^Q$,

$$\begin{aligned}\hat{\gamma}(\varphi_P \circ f) &= \hat{\gamma}((\Gamma \circ \varphi_{PQ})(f))\\ &= (\hat{\gamma} \circ \Gamma \circ \varphi_{PQ})(f)\\ &= (\Gamma \circ \Omega(\gamma) \circ \varphi_{PQ})(f)\\ &= (\Gamma \circ \varphi_{PQ} \circ \gamma)(f)\\ &= (\Gamma \circ \varphi_{PQ})(\gamma(f))\\ &= \varphi_P \circ \gamma(f).\end{aligned}$$

As $\bar{P}^Q$ is join- and meet-dense in $\mathbf{DM}(P)^Q$ (Corollary 2), the uniqueness of $\hat{\gamma}$ follows. □

PROPOSITION 8.8. *Let P be a bounded poset, Q any poset, and $\gamma \in \operatorname{Aut}(P^Q)$. Let $\hat{\gamma}$ be the automorphism of $\mathbf{DM}(P)^Q$ induced by γ. If $\hat{\gamma} \in \operatorname{Im} \nu_{\mathbf{DM}(P),Q}$, then $\gamma \in \operatorname{Im} \nu_{P,Q}$.*

Proof. Without loss of generality, $Q \neq \emptyset$. By assumption, there exist $\mu \in \mathrm{Aut}(\mathbf{DM}(P))$ and $\rho \in \mathrm{Aut}(Q)$ such that

$$\hat{\gamma} = \nu_{\mathbf{DM}(P),Q}(\mu, \rho).$$

Indeed, from Proposition 7, $\mu \in \mathrm{Aut}_{\bar{P}}(\mathbf{DM}(P))$. By Proposition 4, there exists $\lambda \in \mathrm{Aut}(P)$ such that $\mu \circ \varphi_P = \varphi_P \circ \lambda$. We claim that $\gamma = \nu_{P,Q}(\lambda, \rho)$.

For all $f \in P^Q$,

$$\begin{aligned} \varphi_P \circ (\nu_{P,Q}(\lambda, \rho))(f) &= \varphi_P \circ \lambda \circ f \circ \rho^{-1} \\ &= \mu \circ \varphi_P \circ f \circ \rho^{-1} \\ &= (\nu_{\mathbf{DM}(P),Q}(\mu, \rho))(\varphi_P \circ f) \\ &= \hat{\gamma}(\varphi_P \circ f) \\ &= \varphi_P \circ \gamma(f). \end{aligned}$$

As φ_P is an embedding, we have $(\nu_{P,Q}(\lambda, \rho))(f) = \gamma(f)$ for all $f \in P^Q$, so that $\nu_{P,Q}(\lambda, \rho) = \gamma$. □

We conclude the following.

THEOREM 8.9. *Let P be a bounded poset and Q any poset. If $\nu_{\mathbf{DM}(P),Q}$ is surjective, so is $\nu_{P,Q}$.* □

The final result follows easily from Theorem 9 and Theorems 5.6, 6.18, and 7.7.

COROLLARY 8.10. *Let P be a bounded poset such that* **DM**(P) *is directly and exponentially indecomposable. Let Q be a connected non-empty poset. Then*

$$\nu_{P,Q}: \mathrm{Aut}(P) \times \mathrm{Aut}(Q)^{\partial} \to \mathrm{Aut}(P^Q)$$

is an isomorphism of ordered groups if any one of the following holds:

(1) *P is a jm-poset;*
(2) *P is a j-poset and Q is finitely factorable;*
(3) *Q is finite.* □

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