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Priestley powers of lattices and their congruences. A problem of E. T. Schmidt

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For Professor E.T. Schmidt on his sixtieth birthday

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Abstract. Let L be a lattice and M a bounded distributive lattice. Let $\operatorname{Con} L$ denote the congruence lattice of L, $P(M)$ the Priestley dual space of M, and $L^{P(M)}$ the lattice of continuous order-preserving maps from $P(M)$ to L with the discrete topology. It is shown that $\operatorname{Con}(L^{P(M)}) \cong (\operatorname{Con} L)_{\Lambda}^{P(\operatorname{Con} M)}$, the lattice of continuous order-preserving maps from $P(\operatorname{Con} M)$ to $\operatorname{Con} L$ with the Lawson topology. Various other ways of expressing $\operatorname{Con}(L^P)$ as a lattice of continuous functions or semilattice homomorphisms are presented. Indeed, a link is established between semilattice homomorphisms from a semilattice S into a bounded distributive lattice T (or its ideal lattice) and continuous order-preserving maps from $P(T)$ into the ideal lattice of S with the Scott, Lawson, or discrete topology. It is also shown that, in general, $\operatorname{Con}(L^{P(M)}) \ncong (\operatorname{Con} L)^{P(\operatorname{Con} M)}$, solving a problem of E. T. Schmidt (independently solved by Grätzer and Schmidt).

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1. Introduction

A *Priestley power* of a (semi-) lattice L is a (semi-) lattice L^P of continuous order-preserving maps from a Priestley space P to L, where L has the discrete topology (cf. [17], p. 105). (The maps are ordered pointwise.) Every Priestley space arises as the poset of prime filters $P(M)$ of a bounded distributive lattice M, appropriately topologized. Hence Boolean powers ([2], Definition IV.5.3) are a special case. If L and M belong to the category $\mathbf{D}$ of bounded distributive lattices, then $L^{P(M)}$ is the coproduct of L and M in $\mathbf{D}$ ([5], Corollary 2.3; [6], Theorem and Corollary; [24], Theorem).

In [26], the following problem is stated.

Problem. [26]. *If L is a lattice and M a bounded distributive lattice, is the congruence lattice* $\mathrm{Con}(L^{P(M)}) \cong (\mathrm{Con}\, L)^{P(\mathrm{Con}\, M)}$*?*

The problem has been solved in the affirmative for arbitrary L and finite M ([9], Theorem 2.1) as well as for finite L and arbitrary M ([26], Theorem). We solve the problem completely by showing that

$$\mathrm{Con}(L^{P(M)}) \cong (\mathrm{Con}\, L)_{\Lambda}^{P(\mathrm{Con}\, M)},$$

the lattice of continuous order-preserving maps from $P(\mathrm{Con}\, M)$ to $\mathrm{Con}\, L$ with the Lawson topology Λ (Corollary 6.11). We present an example to show that, in general,

$$\mathrm{Con}(L^{P(M)}) \ncong (\mathrm{Con}\, L)^{P(\mathrm{Con}\, M)},$$

(Proposition 7.4). Grätzer and Schmidt have proven that the isomorphism holds if and only if either $\mathrm{Con}\, L$ is finite or M is finite ([15], Theorem 3). Our results were proven independently.

Our approach is to use the results for finite exponents to get the corresponding results for Priestley powers. By [28], Theorem, every Priestley space P is the inverse limit of a filtered system of finite posets Q with the discrete topology. Hence every Priestley power L^P is the filtered limit of lattices L^Q. Using an idea of [23], pp. 98–100, we can capture the congruence lattice of such a limit if we know $\mathrm{Con}(L^Q)$ for every L^Q in the system. By [9], Theorem 2.1, we do. (This approach was also taken in [15], §4, but certain non-trivial steps were passed over without proof.)

We represent various types of posets of continuous order-preserving maps as posets of semilattice homomorphisms (Theorem 3.6, Corollaries 3.7 and 3.8). For example, if S is a semilattice with least element 0 and $T \in \mathbf{D}$, then

$$\mathbf{Slat}(S, T) \cong (S_{\Lambda}^{\sigma\partial})^{P(T)},$$

where S^σ is the ideal lattice of S and $S^{\sigma\partial}$ this lattice ordered by reverse inclusion.

These representations enable us to provide several alternative representations of $\mathrm{Con}(L^P)$. For example,

$$\mathrm{Comp}(L^P) \cong (\mathrm{Comp}\, L)^{\overline{P}}$$

where L is a lattice, P a Priestley space, $\overline{P}$ the same space with the trivial order, and $\mathrm{Comp}\, L$ the semilattice of compact congruences of L (Theorem 5.10). Also

$$\mathrm{Con}(L^P) \cong (\mathrm{Con}\, L)^{\overline{P}}_{\Sigma},$$

the lattice of continuous maps from P to $\mathrm{Con}\, L$ where the latter has the Scott topology Σ (Theorem 6.7). Alternatively, if $M \in \mathbf{D}$, then

$$\mathrm{Con}(L^{P(M)}) \cong \mathbf{Slat}\Big((\mathrm{Comp}\, L, \vee, 0_{\mathrm{Con}\, L}), ({M_{\mathrm{Bool}}}^{\sigma}, \cap, M_{\mathrm{Bool}})\Big),$$

the lattice of semilattice homomorphisms from the $\{0\}$-$\vee$-semilattice $\mathrm{Comp}\, L$ to the $\{1\}$-$\cap$-semilattice ${M_{\mathrm{Bool}}}^{\sigma}$, where M_{Bool} is the minimal Boolean extension of M (Corollary 6.8). Also

$$\mathrm{Con}(L^{P(M)}) \cong \mathbf{Slat}\Big((\mathrm{Comp}\, L, \vee, 0_{\mathrm{Con}\, L}), (\mathrm{Con}\, M, \cap, 1_{\mathrm{Con}\, M})\Big)$$

(Corollary 6.10). These representations enable us to relate special cases of our results to those of [3] and [14] on semilattice homomorphisms between distributive lattices. In particular, we prove that $\mathbf{Slat}(L, L)$ is self-dual for a finite distributive lattice L (Corollary 3.9). Finally, our representations let us construct the example which yields a negative solution to the problem.

2. Notation, definitions, and basic theory

Let us introduce notation and remind ourselves of some definitions and basic results. (See [7], [16], inter alia.) If a poset P has a least element, we denote it 0_P or 0; if it has a greatest element, we denote it 1_P or 1. A poset with 0 and 1 is *bounded*.

Denote the ordinal sum of posets P and Q by $P \oplus Q$. Let $\mathcal{P}(X)$ denote the power set of the set X. Let $\mathbf{1}$ denote the one-one element poset.

Let P be a poset and Q and S subsets. Then $\uparrow_Q S$ denotes

$$\{\, p \in Q \mid s \leq p \text{ for some } s \in S \,\}$$

and $\downarrow_Q S$ denotes

$$\{\, p \in Q \mid s \geq p \text{ for some } s \in S \,\}.$$

We also write $\uparrow S$ for $\uparrow_P S$ and $\downarrow S$ for $\downarrow_P S$. For $s \in P$, we use $\uparrow s$ and $\downarrow s$ for $\uparrow \{s\}$ and $\downarrow \{s\}$, respectively. If $S = \uparrow S$, it is an *up-set*; if $S = \downarrow S$, it is a *down-set*. A non-empty subset D of P is *directed* if every finite subset of D has an upper bound in D. If D has a join it is denoted $\bigsqcup D$. (The special notation, which is standard, serves as a convenient reminder that the set under consideration is directed.) An *ideal* is a directed down-set; the set of all such, ordered by inclusion, is denoted P^σ. A *filtered* subset of P is a directed subset of the poset P^∂ whose order is dual to that of P. A *filter* is an ideal of P^∂. The poset of filters of P is denoted P^π.

An element k of a poset P is *compact* if, for all directed subsets D of P such that $\bigsqcup D$ exists and $p \leq \bigsqcup D$, there exists $d \in D$ such that $k \leq d$. The poset of compact elements is denoted $\kappa(P)$. If P is a complete lattice, an element $k \in P$ is compact if and only if, for all $S \subseteq P$ such that $k \leq \bigvee S$, there exists a finite subset $T \subseteq S$ such that $k \leq \bigvee T$ ([7], Lemma 3.22). An *algebraic lattice* is a complete lattice such that every element is a join of compact elements.

The class of semilattices with neutral element is denoted **Slat**. [The neutral element is 0 for $\vee$-semilattices and 1 for $\wedge$-semilattices ([4], p. 50).] If S and $T \in \mathbf{Slat}$, then $\mathbf{Slat}(S,T)$ denotes the poset of **Slat**-morphisms from S to T ordered pointwise, i.e., for f, $g \in \mathbf{Slat}(S,T)$, $f \leq g$ if $f(s) \leq g(s)$ for all $s \in S$. The subset of **Slat**-morphisms f whose images $\operatorname{Im} f$ are finite is denoted $\mathbf{Slat}^{\mathrm{fin}}(S,T)$. Let **Lat** be the class of lattices. We regard **Slat** and **Lat** as categories with the appropriate morphisms.

A $\vee$-semilattice S with 0 is *distributive* if, whenever a, x, $y \in S$ and

$$a \leq x \vee y,$$

there exist b, $c \in S$ such that $b \leq x$, $c \leq y$, and $a = b \vee c$. Equivalently, S^σ is a distributive lattice. We shall use Stone duality for the class **DSlat** of distributive $\vee$-semilattices with 0 ([13], II.5).

A proper ideal I of $S \in \mathbf{DSlat}$ is *prime* if, whenever a, $b \in S$ and $c \leq a$, b implies $c \in I$ for all $c \in S$, then $a \in I$ or $b \in I$. For all $a \in S$, let

$$\hat{a} := \{\, I \in S^\sigma \mid I \text{ prime}, a \notin I \,\}.$$

Let $\mathcal{S}(S)$ be the set of prime ideals of S with the topology generated by the basis $\{\, \hat{a} \mid a \in S \,\}$. Then $\mathcal{S}(S)$ is the *Stone space* of S.

Given a topological space X, let $\mathcal{O}(X)$ denote the bounded distributive lattice of open sets and $\mathcal{CO}(X)$ the $\cup$-semilattice with least element $\emptyset$ of compact open sets. A topological space is *sober* if every non-empty $\cup$-prime (i.e., $\cup$-irreducible) closed set is the closure of a point. Stone spaces may be abstractly characterized as sober T_0 spaces X such that $\mathcal{CO}(X)$ is a basis. See [29], Lemma 1, Lemma 3, and Satz 4. Indeed, if $S \in \mathbf{DSlat}$, then $\mathcal{CO}\Big(\mathcal{S}(S)\Big) = \{\, \hat{a} \mid a \in S \,\}$. The map $a \mapsto \hat{a}$ ($a \in S$) is an isomorphism from S onto $\mathcal{CO}\Big(\mathcal{S}(S)\Big)$ ([29], pp. 360–361).

Given $L \in \mathbf{Lat}$, let $\operatorname{Con} L$ denote the lattice of congruences of L. It is well-known that $\operatorname{Con} L$ is a distributive algebraic lattice ([1], Theorem II.9.15). For $X \subseteq L \times L$, let

$$\vartheta^L(X) := \bigcap \{\theta \in \operatorname{Con} L \mid X \subseteq \theta \,\}.$$

Let $\operatorname{Comp} L := \kappa(\operatorname{Con} L)$. It is well-known that

$$\operatorname{Comp} L = \left\{ \bigvee_{i=1}^{n} \vartheta^L(a_i, b_i) \;\middle|\; n \geq 0,\, a_i,\, b_i \in L \ (i = 1,\ldots,n) \right\}.$$

If $M \in \mathbf{Lat}$ and $f\colon L \to M$ is a homomorphism, let

$$\operatorname{Comp}(f)\colon \operatorname{Comp}(L) \to \operatorname{Comp}(M)$$

denote the function ([23], p. 98)

$$[\operatorname{Comp}(f)](\theta) := \vartheta^M\Big((f \times f)[\theta]\Big) \qquad (\theta \in \operatorname{Comp} L).$$

If A is an algebraic lattice, then $\Big(\kappa(A), \vee, 0_A\Big) \in \mathbf{Slat}$ and $\Big(\kappa(A)\Big)^\sigma \cong A$ via the map $I \mapsto \bigsqcup I$ ($I \in \kappa(A)^\sigma$) with inverse $a \mapsto {\downarrow_{\kappa(A)}}\, a$ ($a \in A$). Further, if $(S, \vee, 0_S) \in \mathbf{Slat}$, then S^σ is an algebraic lattice and $\kappa(S^\sigma) = \{\, \downarrow s \mid s \in S \,\}$, which is isomorphic to S. (See [10], Corollary 2.) Similarly, if S is a bounded lattice then $\kappa(S^\pi) = \{\, \uparrow s \mid s \in S \,\}$.

If A is an algebraic lattice, the *Scott topology* is the topology

$$\begin{aligned}\Sigma := \{\, U \subseteq A \mid U = {\uparrow} U \text{ and for all directed } D \subseteq A,\\ \bigsqcup D \in U \implies D \cap U \neq \emptyset \,\}.\end{aligned}$$

The *Lawson topology* is the topology Λ on A generated by the subbasis

$$\Sigma \cup \{\, A \setminus \uparrow a \mid a \in A \,\}.$$

(See [12], pp. 99, 144.)

If P and Q are ordered spaces and Q has topology τ, Q^P_τ is the poset of continuous order-preserving maps from P to Q ordered pointwise; Q^P is Q^P_τ where τ is the discrete topology.

An ordered space P is *totally order-disconnected* if, for all p, $q \in P$ such that $p \not\leq q$, there exists a clopen up-set $U \subseteq P$ such that $p \in U$, $q \notin U$. A *Priestley space* is a compact totally order-disconnected ordered space. Let $\mathbf{P}$ denote the category of Priestley spaces with continuous order-preserving maps. Let $\mathbf{P}^{\mathrm{fin}}$ denote the full subcategory of finite Priestley spaces. By the proofs of [12], Theorems III.1.9 and III.1.10, an algebraic lattice with the Lawson topology is a Priestley space.

If P is an ordered space, let $D(P)$ denote the set of clopen up-sets of P; let $\mathcal{U}(P)$ denote the set of open up-sets.

Let $\mathbf{D}$ denote the category of bounded distributive lattices with $\{0,1\}$-homomorphisms (homomorphisms preserving 0 and 1). For $L \in \mathbf{D}$, let $P(L)$ denote the Priestley space of prime filters of L, appropriately topologized. Let $\mathcal{J}(L)$ denote the poset of join-irreducible elements of L. For $a \in L$, let

$$\rho_L(a) := \{\, F \in P(L) \mid a \in F \,\}.$$

It is well-known that $\mathbf{D}$ and $\mathbf{P}$ are dually equivalent categories, $D(-)$ and $P(-)$ being the functors yielding the dual equivalence. We shall identify a lattice with the clopen up-sets of its Priestley dual space and shall not differentiate between the abstract and concrete forms of the lattice. For the details of Priestley duality, see [20], [21].

If $L \in \mathbf{D}$, there is an isomorphism from L^σ to $\mathcal{U}\Big(P(L)\Big)$. Refer to [22], §8; see also [7], 10.24.

If $P \in \mathbf{P}$, let $\overline{P}$ denote the trivially ordered Priestley space with the same topology as P. We denote the minimal Boolean extension of $L \in \mathbf{D}$ by L_{Bool}. See [1], Definition V.4.5, [21], §6.

If $L \in \mathbf{D}$, then $\operatorname{Con} L$ is dually isomorphic to the lattice of closed subsets of $P(L)$ ([7], 10.27).

Let P be a set, Π, Π_0 partitions of P. Let $\nu_\Pi\colon P \to \Pi$ be the map assigning each element of P its equivalence class. The set of partitions of P is ordered as follows: $\Pi \leq \Pi_0$ if every equivalence class of Π is contained in some equivalence class of Π_0. If $\Pi \leq \Pi_0$, let

$$\nu_{\Pi,\Pi_0}\colon \Pi \to \Pi_0$$

be the map assigning each equivalence class of Π the unique equivalence class of Π_0 containing it.

Given a partition $\Pi := \{V_i\}_{i \in I}$ of a poset P into equivalence classes indexed by a set I, we define a quasiorder $\leq_\Pi$ on Π as follows. Let $\leq_\Pi$ be the transitive closure of the relation $\preceq_\Pi$ defined in this way: $V_i \preceq_\Pi V_j$ if $p \leq q$ for some $p \in V_i$, $q \in V_j$ $(i,\, j \in I)$.

If $P \in \mathbf{P}$, denote by $\mathcal{E}_P$ the ordered set of partitions Π of P into open equivalence classes such that $(\Pi, \leq_\Pi)$ is partially ordered. Regard Π as a space with the discrete topology. The same partition Π with the antichain ordering is denoted $\overline{\Pi}$. Let $\overline{\mathcal{E}_P} := \{\, \overline{\Pi} \mid \Pi \in \mathcal{E}_P \,\}$.

Let $P \in \mathbf{P}$, $M \in \mathbf{Lat} \cup \mathbf{Slat}$. For every $\Pi \in \mathcal{E}_P$, let $\mu^M_\Pi \colon M^\Pi \to M^P$ be defined by $\mu^M_\Pi(f) := f \circ \nu_\Pi$ $(f \in M^\Pi)$.

For Π, $\Pi_0 \in \mathcal{E}_P$ such that $\Pi \leq \Pi_0$, let

$$\mu^M_{\Pi,\Pi_0} \colon M^{\Pi_0} \to M^\Pi$$

be defined by

$$\mu^M_{\Pi,\Pi_0}(f) := f \circ \nu_{\Pi,\Pi_0} \qquad (f \in M^{\Pi_0}).$$

For $L \in \mathbf{Lat}$, $P \in \mathbf{P}^{\mathrm{fin}}$, and $p \in P$, denote by χ_p the kernel of the p-th projection of L^P onto L. Define

$$\Gamma'_P \colon \mathrm{Con}(L^P) \to (\mathrm{Con}\, L)^{\overline{P}}$$

as follows: for $\theta \in \mathrm{Con}(L^P)$ and $p \in P$, let

$$[\Gamma'_P(\theta)](p) := \{\, (a,b) \in L \times L \mid (f,g) \in \theta \vee \chi_p \text{ for all } f,\, g \in L^P \text{ such that } f(p) = a,\ g(p) = b \,\}.$$

Define

$$\Gamma_P \colon \mathrm{Comp}(L^P) \to (\mathrm{Comp}\, L)^{\overline{P}}$$

by $\Gamma_P(\theta) := \Gamma'_P(\theta)$ for all $\theta \in \mathrm{Comp}(L^P)$. Define

$$\Delta'_P \colon (\mathrm{Con}\, L)^{\overline{P}} \to \mathrm{Con}(L^P)$$

as follows: for $F \in (\mathrm{Con}\, L)^{\overline{P}}$, let

$$\Delta'_P(F) := \{\, (f,g) \in L^P \times L^P \mid \big(f(p), g(p)\big) \in F(p) \text{ for all } p \in P \,\}.$$

Define

$$\Delta_P \colon (\mathrm{Comp}\, L)^{\overline{P}} \to \mathrm{Comp}(L^P)$$

by $\Delta_P(F) := \Delta'_P(F)$ for all $F \in (\mathrm{Comp}\, L)^{\overline{P}}$. (That the above functions are well defined will be shown in Proposition 5.7.)

If $L \in \mathbf{Lat}$ and $P \in \mathbf{P}^{\text{fin}}$, a, $b \in L$, and $p_0 \in P$, define $m_P(a, b, p_0)\colon P \to L$ for all $p \in P$ as follows:

$$[m_P(a,b,p_0)](p) := \begin{cases} a & \text{if } p = p_0, \\ a \vee b & \text{if } p > p_0, \\ a \wedge b & \text{else.} \end{cases}$$

Finally, we remind ourselves of basic categorical notions. Let $\mathbf{C}$ be a category and F a filtered poset. Let $(C_i)_{i \in F}$ be a family of objects of $\mathbf{C}$ and

$$(f_{ij}\colon C_j \to C_i)_{\substack{i,j \in F \\ i \leq j}}$$

a family of morphisms with the following properties:
(1) $f_{ii} = \mathrm{id}(C_i)$ for all $i \in F$;
(2) $f_{ij} \circ f_{jk} = f_{ik}$ for all i, j, $k \in F$ such that $i \leq j \leq k$.
Then

$$\mathcal{S} := \Big((C_i)_{i \in F}, (f_{ij}\colon C_j \to C_i)_{\substack{i,j \in F \\ i \leq j}}\Big)$$

is a *filtered system* in $\mathbf{C}$. Assume $C \in \mathbf{C}$ and $(f_i\colon C_i \to C)_{i \in F}$ is a family of morphisms such that $i \leq j$ implies $f_i \circ f_{ij} = f_j$ $(i, j \in F)$. Then

$$\Big(C, (f_i\colon C_i \to C)_{i \in F}\Big)$$

is *compatible* with the filtered system $\mathcal{S}$. Assume $\Big(C, (f_i\colon C_i \to C)_{i \in F}\Big)$ also has the property that, for any $\Big(C', (f'_i\colon C_i \to C')_{i \in F}\Big)$ compatible with $\mathcal{S}$, there is a unique morphism $f\colon C \to C'$ such that $f \circ f_i = f'_i$ for all $i \in F$. Then $\Big(C, (f_i\colon C_i \to C)_{i \in F}\Big)$ is a *filtered limit* of $\mathcal{S}$.

Let $(D_i)_{i \in F}$ be a family of objects of $\mathbf{C}$ and

$$(g_{ij}\colon D_i \to D_j)_{\substack{i,j \in F \\ i \leq j}}$$

a family of morphisms with the following properties:
(1) $g_{ii} = \mathrm{id}(D_i)$ for all $i \in F$;
(2) $g_{jk} \circ g_{ij} = g_{ik}$ for all i, j, $k \in F$ such that $i \leq j \leq k$.
Then

$$\mathcal{T} := \Big((D_i)_{i \in F}, (g_{ij}\colon D_i \to D_j)_{\substack{i,j \in F \\ i \leq j}}\Big)$$

is an *inverse system* in $\mathbf{C}$. Assume $D \in \mathbf{C}$ and $(g_i\colon D \to D_i)_{i \in F}$ is a family of morphisms such that $i \leq j$ implies $g_{ij} \circ g_i = g_j$ $(i, j \in F)$. Then

$$\Big(D, (g_i\colon D \to D_i)_{i \in F}\Big)$$

is *compatible* with the inverse system $\mathcal{T}$. Assume $\Bigl(D, (g_i\colon D \to D_i)_{i\in F}\Bigr)$ also has the property that, for any $\Bigl(D', (g_i'\colon D' \to D_i)_{i\in F}\Bigr)$ compatible with $\mathcal{T}$, there is a unique morphism $g\colon D' \to D$ such that $g_i \circ g = g_i'$ for all $i \in F$. Then $\Bigl(D, (g_i\colon D \to D_i)_{i\in F}\Bigr)$ is an *inverse limit* of $\mathcal{T}$.

A result will be referred to without a section number in the section in which it appears.

3. Continuous function duals of semilattice homomorphisms

In this section we show how various posets of **Slat**-morphisms may be viewed as posets of continuous order-preserving maps from a Priestley space into an ideal lattice with an appropriate topology (Theorem 6, Corollary 7, and Corollary 8). We then show how *Priestley relations*, introduced in [3] as the duals of $\{0\}$-$\vee$-homomorphisms between bounded distributive lattices under Priestley duality, correspond naturally with such function spaces (Proposition 12).

Lemma 3.1. *Let A be an algebraic lattice. The family $\{\, \uparrow k \mid k \in \kappa(A) \,\}$ is closed under finite (including empty) intersections and is a basis for Σ.*

Hence $\{\, \uparrow k \mid k \in \kappa(A) \,\} \cup \{\, A \setminus \uparrow a \mid a \in A \,\}$ is a subbasis for Λ.

Proof. See [12], Corollary II.1.15. ∎

Lemma 3.2. *Let A be an algebraic lattice and let $P \in \mathbf{P}$ and $p \in P$. Let*

$$g \in \mathbf{Slat}\Bigl(\bigl(\kappa(A), \vee, 0_A\bigr), \bigl(\mathcal{U}(P), \cap, P\bigr)\Bigr).$$

Then $\{\, k \in \kappa(A) \mid p \in g(k) \,\} \in \kappa(A)^\sigma$. Hence for all $k_0 \in \kappa(A)$,

$$k_0 \le \bigsqcup \{\, k \in \kappa(A) \mid p \in g(k) \,\} \iff p \in g(k_0).$$

Proof. Let $I := \{\, k \in \kappa(A) \mid p \in g(k) \,\}$. As $g(0_A) = P$, we have $0_A \in I$.

If $k_0 \in \kappa(A)$, $k \in I$, and $k_0 \le k$, then $p \in g(k) \subseteq g(k_0)$, so $k_0 \in I$.

If k_0, $k_1 \in I$, then $p \in g(k_0) \cap g(k_1) = g(k_0 \vee k_1)$, so $k_0 \vee k_1 \in I$. Therefore $I \in \kappa(A)^\sigma$.

By the isomorphism between $\kappa(A)^\sigma$ and A of §2, for all $k \in \kappa(A)$, $k \le \bigsqcup I$ if and only if $k \in I$. ∎

Lemma 3.3. *Let A be an algebraic lattice and let $P \in \mathbf{P}$. Let $f\colon P \to A$ be a map. Assume*

$$\{\, f^{-1}(\uparrow k) \mid k \in \kappa(A)\,\}$$

is finite. Then for all $a \in A$, there exists $k \in \downarrow_{\kappa(A)} a$ such that

$$f^{-1}(\uparrow a) = f^{-1}(\uparrow k).$$

Proof. Let $a \in A$. Let $k_0 \in \downarrow_{\kappa(A)} a$ be such that $f^{-1}(\uparrow k_0)$ is minimal in

$$\{\, f^{-1}(\uparrow k) \mid k \in \downarrow_{\kappa(A)} a\,\}.$$

Then for all $k \in \kappa(A)$ such that $k_0 \leq k \leq a$, we have $f^{-1}(\uparrow k) = f^{-1}(\uparrow k_0)$. Therefore

$$\begin{aligned} f^{-1}(\uparrow a) &= f^{-1}\Bigg(\bigcap_{k\in\kappa(A)\cap\downarrow a} \uparrow k\Bigg) = \bigcap_{k\in\kappa(A)\cap\downarrow a} f^{-1}(\uparrow k) \\ &= \bigcap_{\substack{k\in\kappa(A)\\ k_0\leq k\leq a}} f^{-1}(\uparrow k) = f^{-1}(\uparrow k_0). \end{aligned}$$

■

Lemma 3.4. *Let A be an algebraic lattice and let $P \in \mathbf{P}$. Let $f \in A^P_\Sigma$. The following are equivalent:*
(1) *$f \in A^P_\Lambda$;*
(2) *for all $k \in \kappa(A)$, $f^{-1}(\uparrow k)$ is closed.*
In either case, for all $k \in \kappa(A)$, $f^{-1}(\uparrow k) \in D(P)$.

Proof. See [16], §V.

■

Lemma 3.5. *Let A be an algebraic lattice and let $P \in \mathbf{P}$. Let $f \in A^P_\Lambda$. The following are equivalent:*
(1) *$f \in A^P$;*
(2) *$\operatorname{Im} f$ is finite;*
(3) *$\{\, f^{-1}(\uparrow k) \mid k \in \kappa(A)\,\}$ is finite.*

Proof. (1)$\Longrightarrow$(2). The implication holds because P is compact and f is a continuous map into a space with the discrete topology, so $\operatorname{Im} f$ is compact and hence finite.

(2)$\Longrightarrow$(3). For all $k \in \kappa(A)$,

$$f^{-1}(\uparrow k) = \bigcup\{\, f^{-1}(a) \mid a \in \operatorname{Im} f \text{ and } k \leq a \,\}$$

so there are at most 2^n elements in $\{\, f^{-1}(\uparrow k) \mid k \in \kappa(A) \,\}$ where n is the size of $\operatorname{Im} f$.

(3)$\Longrightarrow$(1). By Lemmas 3 and 4, $\{\, f^{-1}(\uparrow a) \mid a \in A \,\}$ is finite and $f^{-1}(\uparrow a)$ is clopen for all $a \in A$. Let $a \in A$. Then

$$\{\, f^{-1}(\uparrow b) \mid b \in A \text{ and } a < b \,\} = \{\, f^{-1}(\uparrow b_i) \mid i = 1, \ldots, n \,\}$$

for some $n \geq 0$, $b_i \in A$ such that $a < b_i$ $(i = 1, \ldots, n)$. Then

$$\begin{aligned} f^{-1}(a) &= f^{-1}(\uparrow a) \setminus \Big(\bigcup \Big\{\, f^{-1}(\uparrow b) \;\Big|\; b \in A \text{ where } a < b \,\Big\} \Big) \\ &= f^{-1}(\uparrow a) \setminus \Big(\bigcup_{i=1}^{n} f^{-1}(\uparrow b_i) \Big), \end{aligned}$$

which is open. Hence $f \in A^P$. ■

Theorem 3.6. *Let A be an algebraic lattice and let $P \in \mathbf{P}$. By $\kappa(A)$ and $\mathcal{U}(P)$ we shall mean the objects $\big(\kappa(A), \vee, 0_A\big)$ and $\big(\mathcal{U}(P), \cap, P\big)$ of* **Slat**. *Define a map*

$$\Psi \colon A^P_\Sigma \to \mathbf{Slat}\Big(\kappa(A), \mathcal{U}(P)\Big)$$

as follows: for $f \in A^P_\Sigma$ and $k \in \kappa(A)$, let

$$[\Psi(f)](k) := f^{-1}(\uparrow k).$$

Define a map

$$\Phi \colon \mathbf{Slat}\Big(\kappa(A), \mathcal{U}(P)\Big) \to A^P_\Sigma$$

as follows: for $g \in \mathbf{Slat}\big(\kappa(A), \mathcal{U}(P)\big)$ and $p \in P$, let

$$[\Phi(g)](p) := \bigsqcup\{\, k \in \kappa(A) \mid p \in g(k) \,\}.$$

Then Ψ and Φ are mutually-inverse order-isomorphisms. The restriction of Ψ to A^P_Λ maps onto $\mathbf{Slat}\big(\kappa(A), D(P)\big)$. The restriction of Ψ to A^P maps onto $\mathbf{Slat}^{\text{fin}}\big(\kappa(A), D(P)\big)$.

Proof. Let $f \in A^P_\Sigma$, k, k_1, $k_2 \in \kappa(A)$. As f is continuous and order-preserving, by Lemma 1 $f^{-1}(\uparrow k) \in \mathcal{U}(P)$. Also $f^{-1}(\uparrow 0_A) = f^{-1}(A) = P$.

Finally $f^{-1}\Big(\uparrow(k_1 \vee k_2)\Big) = f^{-1}\Big((\uparrow k_1) \cap (\uparrow k_2)\Big) = f^{-1}(\uparrow k_1) \cap f^{-1}(\uparrow k_2)$. So the map $k \mapsto f^{-1}(\uparrow k)$ $[k \in \kappa(A)]$ is in $\mathbf{Slat}\Big(\kappa(A), \mathcal{U}(P)\Big)$. Thus Ψ is well-defined.

Let f_1, $f_2 \in A^P_\Sigma$ be such that $f_1 \leq f_2$. For $k \in \kappa(A)$,

$$\begin{aligned}[\Psi(f_1)](k) &= f_1^{-1}(\uparrow k) = \{\, p \in P \mid k \leq f_1(p) \,\} \\ &\subseteq \{\, p \in P \mid k \leq f_2(p) \,\} = f_2^{-1}(\uparrow k) = [\Psi(f_2)](k).\end{aligned}$$

Hence $\Psi(f_1) \leq \Psi(f_2)$, so Ψ is order-preserving.

Let $g \in \mathbf{Slat}\Big(\kappa(A), \mathcal{U}(P)\Big)$ and $p_0 \in P$. By Lemma 2,

$$\{\, k \in \kappa(A) \mid p_0 \in g(k) \,\}$$

is directed. Let $k_0 \in \kappa(A)$ be such that $\bigsqcup\{\, k \in \kappa(A) \mid p_0 \in g(k) \,\} \in \uparrow k_0$. By Lemma 2, $p_0 \in g(k_0)$. As $g(k_0)$ is open, we conclude that the map

$$p \mapsto \bigsqcup\{\, k \in \kappa(A) \mid p \in g(k) \,\} \qquad (p \in P)$$

is continuous from P to A with the Scott topology, by Lemma 1.

Let $p_1, p_2 \in P$ be such that $p_1 \leq p_2$. Let $k_0 \in \kappa(A)$ be such that $p_1 \in g(k_0)$. Then $p_2 \in g(k_0)$, because $g(k_0)$ is an up-set. Therefore

$$\bigsqcup\{\, k \in \kappa(A) \mid p_1 \in g(k) \,\} \leq \bigsqcup\{\, k \in \kappa(A) \mid p_2 \in g(k) \,\}.$$

We conclude that the map $p \mapsto \bigsqcup\{\, k \in \kappa(A) \mid p \in g(k) \,\}$ is order-preserving. Therefore Φ is well-defined.

Let g_1, $g_2 \in \mathbf{Slat}\Big(\kappa(A), \mathcal{U}(P)\Big)$ be such that $g_1 \leq g_2$ and let $p \in P$. For $k \in \kappa(A)$, $p \in g_1(k)$ implies $p \in g_2(k)$, so

$$\begin{aligned}[\Phi(g_1)](p) &= \bigsqcup\{\, k \in \kappa(A) \mid p \in g_1(k) \,\} \\ &\leq \bigsqcup\{\, k \in \kappa(A) \mid p \in g_2(k) \,\} = [\Phi(g_2)](p).\end{aligned}$$

Hence $\Phi(g_1) \leq \Phi(g_2)$, so Φ is order-preserving.

Let $f \in A^P_\Sigma$. For $p_0 \in P$,

$$\begin{aligned}[(\Phi \circ \Psi)(f)](p_0) &= \bigsqcup\{\, k \in \kappa(A) \mid p_0 \in [\Psi(f)](k) \,\} \\ &= \bigsqcup\{\, k \in \kappa(A) \mid p_0 \in f^{-1}(\uparrow k) \,\} \\ &= \bigsqcup\{ k \in \kappa(A) \mid k \leq f(p_0) \,\} = f(p_0),\end{aligned}$$

so $(\Phi \circ \Psi)(f) = f$. That is, $\Phi \circ \Psi = \mathrm{id}(A^P_\Sigma)$.

Now let $g \in \mathbf{Slat}\Big(\kappa(A), \mathcal{U}(P)\Big)$. For $k_0 \in \kappa(A)$,

$$\begin{aligned}[(\Psi \circ \Phi)(g)](k_0) &= [\Phi(g)]^{-1}(\uparrow k_0) = \{\, p \in P \mid k_0 \le [\Phi(g)](p) \,\} \\ &= \Big\{\, p \in P \;\Big|\; k_0 \le \bigsqcup \{\, k \in \kappa(A) \mid p \in g(k) \,\} \Big\}.\end{aligned}$$

By Lemma 2, we have

$$[(\Psi \circ \Phi)(g)](k_0) = \{\, p \in P \mid p \in g(k_0) \,\} = g(k_0),$$

so $(\Psi \circ \Phi)(g) = g$. That is, $\Psi \circ \Phi = \mathrm{id}\Big[\mathbf{Slat}\Big(\kappa(A), \mathcal{U}(P)\Big)\Big]$. Therefore, Ψ and Φ are mutually-inverse order-isomorphisms.

Let $f \in A^P_\Sigma$. By Lemma 4, $f \in A^P_\Lambda$ if and only if

$$\Psi(f) \in \mathbf{Slat}\Big(\kappa(A), D(P)\Big).$$

By Lemma 5, $f \in A^P$ if and only if $\mathrm{Im}\,\Psi(f)$ is finite and

$$\Psi(f) \in \mathbf{Slat}\Big(\kappa(A), D(P)\Big).$$

■

The next corollary follows from Theorem 6 using the **D-P** dictionary for ideals mentioned in §2. It explains the "curious duality" behind the representation of modular lattices of the form M_3^P, where M_3 is the five-element non-distributive modular lattice and P a finite poset ([25], §1, Construction 1).

Corollary 3.7. *Let* $(S, \vee, 0_S) \in \mathbf{Slat}$, $T \in \mathbf{D}$. *We regard* T *and* T^σ *as the objects* $(T, \wedge, 1_T)$ *and* $(T^\sigma, \cap, T)$ *of* $\mathbf{Slat}$, *respectively. Let* $\varphi\colon \kappa(T^\sigma) \cong T$ *be the isomorphism* $\varphi(\downarrow t) = t$ $(t \in T)$. *Define a map*

$$\Psi\colon (S^\sigma_\Sigma)^{P(T)} \to \mathbf{Slat}(S, T^\sigma)$$

as follows: for $f \in (S^\sigma_\Sigma)^{P(T)}$ *and* $s \in S$, *let*

$$[\Psi(f)](s) := \{\, t \in T \mid s \in \bigcap f[\rho_T(t)] \,\}.$$

Define a map

$$\Phi\colon \mathbf{Slat}(S, T^\sigma) \to (S^\sigma_\Sigma)^{P(T)}$$

as follows: for $g \in \mathbf{Slat}(S, T^\sigma)$ *and* $F \in P(T)$*, let*

$$[\Phi(g)](F) := \{\, s \in S \mid F \cap g(s) \neq \emptyset \,\}.$$

Then Ψ *and* Φ *are mutually-inverse order-isomorphisms.*

Define $\Psi' \colon (S^\sigma_\Lambda)^{P(T)} \to \mathbf{Slat}(S, T)$ *as follows: for* $f \in (S^\sigma_\Lambda)^{P(T)}$ *and* $s \in S$*, let*

$$[\Psi'(f)](s) := \varphi\Big[\Big(\Psi(f)\Big)(s)\Big].$$

Define $\Phi' \colon \mathbf{Slat}(S, T) \to (S^\sigma_\Lambda)^{P(T)}$ *as follows: for* $g \in \mathbf{Slat}(S, T)$ *and* $F \in P(T)$*, let*

$$[\Phi'(g)](F) := g^{-1}(F).$$

Then Ψ' *and* Φ' *are mutually-inverse order-isomorphisms. The restriction of* Ψ' *to* $(S^\sigma)^{P(T)}$ *maps onto* $\mathbf{Slat}^{\mathrm{fin}}(S, T)$*.*

By reversing the order of T, we get the following.

Corollary 3.8. *Let* $(S, \vee, 0_S) \in \mathbf{Slat}$*,* $T \in \mathbf{D}$*. We regard* T *as the object* $(T, \vee, 0_T)$ *of* $\mathbf{Slat}$*. Let* $\varphi \colon \kappa(T^\pi) \to T$ *be the dual-isomorphism* $\varphi({\uparrow}\, t) = t$*.*

Define the map $\Psi' \colon (S^{\sigma\partial}_\Lambda)^{P(T)} \to \mathbf{Slat}(S, T)$ *as follows: for* $f \in (S^{\sigma\partial}_\Lambda)^{P(T)}$ *and* $s \in S$*, let*

$$[\Psi'(f)](s) := \varphi\Big(\{\, t \in T \mid s \in \bigcap f[P(T) \setminus \rho_T(t)] \,\}\Big).$$

Define a map

$$\Phi' \colon \mathbf{Slat}(S, T) \to (S^{\sigma\partial}_\Lambda)^{P(T)}$$

as follows: for $g \in \mathbf{Slat}(S, T)$ *and* $F \in P(T)$*, let*

$$[\Phi'(g)](F) := g^{-1}(T \setminus F).$$

Then Ψ' *and* Φ' *are mutually-inverse order-isomorphisms. The restriction of* Ψ' *to* $(S^{\sigma\partial})^{P(T)}$ *maps onto* $\mathbf{Slat}^{\mathrm{fin}}(S, T)$*.*

It has been shown that if L is a finite lattice, then $\mathbf{Slat}(L, L) \in \mathbf{D}$ if and only if $L \in \mathbf{D}$ (see [14], Theorem 3). Indeed, if $L \in \mathbf{D}$ and L is finite, [14], Lemma 1 states that $L^{\mathcal{J}(L)} \cong \mathbf{Slat}(L, L)$. We also have the following

Corollary 3.9. *Let $L \in \mathbf{D}$ be finite. Then*

$$L^{\mathcal{J}(L)} \cong \Big(\mathbf{Slat}(L, L)\Big)^{\partial}.$$

Therefore $\mathbf{Slat}(L, L)$ *is self-dual.*

Proof. As L is finite, $L^{\sigma} \cong L$ and $P(L) = \{\, \uparrow j \mid j \in \mathcal{J}(L) \,\} \cong \mathcal{J}(L)^{\partial}$. ∎

Indeed, $\mathbf{Slat}(L, L)$ is the coproduct of L and L^{∂} in $\mathbf{D}$ for a finite distributive lattice L. (See [5], Corollary 2.3; [6], Theorem and Corollary; and [24], Theorem.)

Under Priestley duality, continuous order-preserving maps between Priestley spaces P and Q correspond to $\{0, 1\}$-preserving homomorphisms between $D(Q)$ and $D(P)$. In [3], $\{0\}$-$\vee$-homomorphisms were shown to correspond to certain relations between P and Q.

Let P, $Q \in \mathbf{P}$; let $R \subseteq P \times Q$. For $p \in P$, $R(p) := \{\, q \in Q \mid (p, q) \in R \,\}$. For $V \subseteq Q$, $R^{-1}(V) := \{\, p \in P \mid R(p) \cap V \neq \emptyset \,\}$. The relation R is a *Priestley relation* if

(1) $R(p)$ is a closed down-set of Q for all $p \in P$;
(2) $R^{-1}(V) \in D(P)$ for all $V \in D(Q)$.

Let $\mathcal{R}(P, Q)$ denote the set of Priestley relations from P to Q.

For $R \in \mathcal{R}(P, Q)$, let $R^{*}\colon D(Q) \to D(P)$ be the function $R^{*}(V) = R^{-1}(V)$ $\Big(V \in D(Q)\Big)$. By [3], Lemma 1.5, it is a $\{0\}$-$\vee$-homomorphism. Indeed, the map

$$R \mapsto R^{*} \qquad [R \in \mathcal{R}(P, Q)]$$

is a bijection between $\mathcal{R}(P, Q)$ and $\mathbf{Slat}\Big(D(Q), D(P)\Big)$ (where we regard $D(P)$ and $D(Q)$ as $\{\emptyset\}$-$\cup$-semilattices).

We shall turn $\mathcal{R}(P, Q)$ into a poset as follows: for R, $S \in \mathcal{R}(P, Q)$, $R \leq S$ if and only if $R(p) \subseteq S(p)$ for all $p \in P$.

Lemma 3.10. *Let P, $Q \in \mathbf{P}$, $R \in \mathcal{R}(P, Q)$. Then for all $p \in P$,*

$$Q \setminus R(p) = \bigcup \{\, V \in D(Q) \mid p \notin R^{*}(V) \,\}.$$

Proof. For all $p \in P$, the set $Q \setminus R(p) \in \mathcal{U}(Q)$. By the isomorphism of §2,

$$\begin{aligned} Q \setminus R(p) &= \bigcup \{\, V \in D(Q) \mid V \subseteq Q \setminus R(p) \,\} \\ &= \bigcup \{\, V \in D(Q) \mid R(p) \subseteq Q \setminus V \,\} \\ &= \bigcup \{\, V \in D(Q) \mid p \notin R^{-1}(V) \,\}. \end{aligned}$$

∎

Lemma 3.11. *Let P, $Q \in \mathbf{P}$. The map*

$$R \mapsto R^* \qquad [R \in \mathcal{R}(P,Q)]$$

from $\mathcal{R}(P,Q)$ to $\mathbf{Slat}\Big[\Big(D(Q),\cup,\emptyset\Big),\Big(D(P),\cup,\emptyset\Big)\Big]$ is an order-isomorphism.

Proof. Let R, $S \in \mathcal{R}(P,Q)$. First assume $R \subseteq S$. Then for all $V \in D(Q)$

$$\begin{aligned} R^*(V) = R^{-1}(V) &= \{\, p \in P \mid R(p) \cap V \neq \emptyset \,\} \\ &\subseteq \{\, p \in P \mid S(p) \cap V \neq \emptyset \,\} = S^{-1}(V) = S^*(V). \end{aligned}$$

Therefore $R^* \leq S^*$. Hence the map is order-preserving.

Now assume $R^* \leq S^*$. By Lemma 10, for all $p \in P$,

$$\begin{aligned} Q \setminus S(p) &= \bigcup \{\, V \in D(Q) \mid p \notin S^*(V) \,\} \\ &\subseteq \bigcup \{\, V \in D(Q) \mid p \notin R^*(V) \,\} = Q \setminus R(p), \end{aligned}$$

so that $R(p) \subseteq S(p)$. Therefore $R \subseteq S$. Hence the map is an order-embedding.

As the map is onto, it is an order-isomorphism. ∎

Now we establish the connection between our function space representation of **Slat**-morphisms and Priestley relations.

Proposition 3.12. *Let P, $Q \in \mathbf{P}$. We regard $D(P)$ and $D(Q)$ as $\{\emptyset\}$-$\cup$-semilattices. Define*

$$\theta \colon \mathcal{R}(P,Q) \to \mathcal{U}(Q)_\Lambda^{P^\partial}$$

as follows: for $R \in \mathcal{R}(P,Q)$ and $p \in P$, let

$$[\theta(R)](p) := Q \setminus R(p).$$

Define

$$\Psi' \colon \mathcal{U}(Q)_\Lambda^{P^\partial} \to \mathbf{Slat}\Big(D(Q), D(P)\Big)$$

as follows:

$$[\Psi'(f)](V) := P \setminus f^{-1}(\uparrow_{\mathcal{U}(Q)} V) \qquad [f \in \mathcal{U}(Q)_\Lambda^{P^\partial},\ V \in D(Q)].$$

Define

$$\Phi' \colon \mathbf{Slat}\Big(D(Q), D(P)\Big) \to \mathcal{U}(Q)_\Lambda^{P^\partial}$$

as follows: for $g \in \mathbf{Slat}\Big(D(Q), D(P)\Big)$ *and* $p \in P$, *let*

$$[\Phi'(g)](p) := \bigcup \{\, V \in D(Q) \mid p \notin g(V) \,\}.$$

Then:

(1) θ *is a dual-isomorphism;*

(2) Ψ' *and* Φ' *are mutually-inverse dual-isomorphisms;*

(3) *for all* $R \in \mathcal{R}(P, Q)$,

$$(\Psi' \circ \theta)(R) = R^*.$$

Proof. By Theorem 6, Ψ' and Φ' are inverse dual-isomorphisms.

For $R \in \mathcal{R}(P, Q)$ and $p \in P$,

$$[\Phi'(R^*)](p) = \bigcup \{\, V \in D(Q) \mid p \notin R^*(V) \,\} = Q \setminus R(p)$$

by Lemma 10. Hence θ is well-defined and $\Phi'(R^*) = \theta(R)$, so

$$(\Psi' \circ \theta)(R) = R^*.$$

■

4. Profinite posets and Priestley powers

In [28], Theorem, it is shown that every Priestley space P is an inverse limit of finite posets with the discrete topology. Although the proof requires minor modifications, the basic idea is to partition the space into finitely many parts and place a partial order on the set of equivalence classes (if possible) so that the natural projection map is continuous and order-preserving. The inverse limit of the filtered system of these ordered partitions will be the original Priestley space (Proposition 6). If one does this same procedure with $\overline{P}$, a priori one will get more partitions. We show, however, that $\overline{P}$ is in fact the inverse limit of the unordered versions of the partitions arising from P (Proposition 7).

If $M \in \mathbf{Lat} \cup \mathbf{Slat}$, then, for each of the above partitions Π of P, one gets a Priestley power M^{Π}, and the filtered limit of these is M^P (Proposition 14). For $\overline{P}$, however, we are not using all the partitions that arise from $\overline{P}$ necessarily, but only those arising from P. While the inverse limit of each filtered system of partitions (the one arising from P, the other from $\overline{P}$) is $\overline{P}$, we must prove that the corresponding filtered limit is $M^{\overline{P}}$ (Proposition 15). We use a lemma, interesting in its own right, to show that if a Priestley space is an inverse limit of two filtered systems of finite antichains, then any partition arising from one system may be refined to yield a partition arising from the other system (Lemma 9).

The first lemmas are easy.

Lemma 4.1. *Let P be a set, Π, Π_0 partitions of P such that $\Pi \leq \Pi_0$. Then*

$$\nu_{\Pi_0} = \nu_{\Pi,\Pi_0} \circ \nu_\Pi.$$

Lemma 4.2. *Let $P \in \mathbf{P}$. Then:*

(1) *every $\Pi \in \mathcal{E}_P$ is a finite poset, the elements of which are non-empty clopen subsets of P;*
(2) *for all $\Pi \in \mathcal{E}_P$, $\nu_\Pi \colon P \to \Pi$ is continuous, order-preserving, and surjective;*
(3) *for all Π, $\Pi_0 \in \mathcal{E}_P$ such that $\Pi \leq \Pi_0$,*

$$\nu_{\Pi,\Pi_0} \colon \Pi \to \Pi_0$$

is order-preserving and surjective;
(4) $\overline{\mathcal{E}_P} \subseteq \mathcal{E}_{\overline{P}}$.

Lemma 4.3. *Let $P \in \mathbf{P}$; let Q be a poset. Let $f \in Q^P$. For each $q \in \operatorname{Im} f$, let $V_q := f^{-1}(q)$; let $\Pi := \{V_q\}_{q \in \operatorname{Im} f}$. Define $g \colon \Pi \to Q$ by $g(V_q) := q$ for all $q \in \operatorname{Im} f$. Then $\Pi \in \mathcal{E}_P$, g is order-preserving, and $f = g \circ \nu_\Pi$.*

Proof. Clearly Π is a partition of P into open subsets. We now prove that the quasiorder $\leq_\Pi$ is antisymmetric. Let q, $r \in \operatorname{Im} f$. Assume that $V_q \leq_\Pi V_r$. Then for some $n \geq 1$, there exist $q_1, \dots, q_n \in \operatorname{Im} f$ such that

$$V_q = V_{q_1} \preceq_\Pi \dots \preceq_\Pi V_{q_n} = V_r.$$

As f is order-preserving,

$$q = q_1 \leq \dots \leq q_n = r,$$

so $q \leq r$. Thus, if q, $r \in \operatorname{Im} f$, $V_q \leq_\Pi V_r$, and $V_r \leq_\Pi V_q$, then $q = r$ and hence $V_q = V_r$. Therefore $\leq_\Pi$ is antisymmetric. We conclude that $\Pi \in \mathcal{E}_P$.

The above shows that g is order-preserving and clearly $f = g \circ \nu_\Pi$. ■

Proposition 4.4. *Let $P \in \mathbf{P}$. The poset $\mathcal{E}_P$ is filtered.*

Proof. If $P = \emptyset$, the partition with no equivalence classes is in $\mathcal{E}_P$. If $P \neq \emptyset$, the partition $\{P\} \in \mathcal{E}_P$. In either case, $\mathcal{E}_P \neq \emptyset$.

Now let $\Pi_1, \Pi_2 \in \mathcal{E}_P$. By Lemma 2 (2),

$$\nu_i := \nu_{\Pi_i} \colon P \to \Pi_i \qquad (i = 1, 2)$$

is continuous and order-preserving. Thus the map $\nu \colon P \to \Pi_1 \times \Pi_2$ defined by $\nu(p) := \big(\nu_1(p), \nu_2(p)\big)$ $(p \in P)$ is a continuous order-preserving map into an ordered space with the discrete topology. For $q \in \operatorname{Im}\nu$, let $V_q := \nu^{-1}(q)$. By Lemma 3,

$$\Pi := \{V_q\}_{q \in \operatorname{Im}\nu} \in \mathcal{E}_P.$$

Clearly $\Pi \leq \Pi_1, \Pi_2$. ■

Lemma 4.5. *Let $P \in \mathbf{P}$.*

(1) *If $U \in D(P)$ is non-empty and proper, then $\{U, P \setminus U\} \in \mathcal{E}_P$.*

(2) *If $p, q \in P$ and $p \not\leq q$, then there exists $\Pi \in \mathcal{E}_P$ such that $\nu_\Pi(p) \not\leq \nu_\Pi(q)$.*

Proof. (1) This part is obvious.

(2) There exists $U \in D(P)$ such that $p \in U$ and $q \in P \setminus U$. Let $\Pi := \{U, P \setminus U\}$. ■

Proposition 4.6 ([28], Theorem)**.** *Let $P \in \mathbf{P}$. Then*

$$\Big((\Pi)_{\Pi \in \mathcal{E}_P}, (\nu_{\Pi_1,\Pi_2} \colon \Pi_1 \to \Pi_2)_{\substack{\Pi_1,\Pi_2 \in \mathcal{E}_P \\ \Pi_1 \leq \Pi_2}}\Big)$$

is an inverse system in $\mathbf{P}$ with inverse limit

$$\Big(P, (\nu_\Pi \colon P \to \Pi)_{\Pi \in \mathcal{E}_P}\Big).$$

Proof. By Proposition 4, $\mathcal{E}_P$ is filtered, and it is clear from Lemma 2 that

$$\mathcal{T} := \Big((\Pi)_{\Pi \in \mathcal{E}_P}, (\nu_{\Pi_1,\Pi_2} \colon \Pi_1 \to \Pi_2)_{\substack{\Pi_1,\Pi_2 \in \mathcal{E}_P \\ \Pi_1 \leq \Pi_2}}\Big)$$

is an inverse system in $\mathbf{P}$. By Lemma 1,

$$\Big(P, (\nu_\Pi \colon P \to \Pi)_{\Pi \in \mathcal{E}_P}\Big).$$

is compatible with $\mathcal{T}$.

Assume

$$\Big(Q, (g_\Pi \colon Q \to \Pi)_{\Pi \in \mathcal{E}_P}\Big)$$

is also compatible with $\mathcal{T}$. We prove that, for each $q \in Q$,

$$\bigcap_{\Pi \in \mathcal{E}_P} \nu_\Pi^{-1}\Big(g_\Pi(q)\Big)$$

is a singleton.

By Lemma 2 (2), $C_\Pi := \nu_\Pi^{-1}\Big(g_\Pi(q)\Big)$ is clopen and non-empty for all $\Pi \in \mathcal{E}_P$. If $\Pi_1,\dots,\Pi_n \in \mathcal{E}_P$ for some $n \geq 0$, there exists $\Pi \in \mathcal{E}_P$ such that $\Pi \leq \Pi_1, \dots, \Pi_n$. As $C_\Pi \neq \emptyset$, there exists $p \in P$ such that $\nu_\Pi(p) = g_\Pi(q)$. By Lemma 1, for $i = 1,\dots,n$,

$$\nu_{\Pi_i}(p) = (\nu_{\Pi,\Pi_i} \circ \nu_\Pi)(p) = (\nu_{\Pi,\Pi_i} \circ g_\Pi)(q) = g_{\Pi_i}(q)$$

by compatibility, so

$$p \in \bigcap_{i=1}^{n} C_{\Pi_i}.$$

By compactness,

$$\bigcap_{\Pi \in \mathcal{E}_P} C_\Pi \neq \emptyset.$$

If p, $p' \in P$ and $p \neq p'$, by Lemma 5 (2) there exists $\Pi \in \mathcal{E}_P$ such that $\nu_\Pi(p) \neq \nu_\Pi(p')$. Hence

$$\bigcap_{\Pi \in \mathcal{E}_P} C_\Pi$$

contains a unique element $g(q)$.

We prove that $g \colon Q \to P$ is continuous and order-preserving. Let $U \in D(P)$ be non-empty and proper. Let $p \in U$. Then $\Pi := \{U, P \setminus U\} \in \mathcal{E}_P$ by Lemma 5 (1) and $g_\Pi^{-1}\Big(\nu_\Pi(p)\Big) = g_\Pi^{-1}(\{U\}) \in D(Q)$. We have

$$\begin{aligned}
\nu_\Pi \circ g = g_\Pi &\Longrightarrow g^{-1} \circ \nu_\Pi^{-1} = g_\Pi^{-1} \\
&\Longrightarrow g^{-1} \circ \nu_\Pi^{-1} \circ \nu_\Pi = g_\Pi^{-1} \circ \nu_\Pi \\
&\Longrightarrow g^{-1}(U) = (g^{-1} \circ \nu_\Pi^{-1} \circ \nu_\Pi)(p) = (g_\Pi^{-1} \circ \nu_\Pi)(p) \in D(Q).
\end{aligned}$$

Hence $g \colon Q \to P$ is order-preserving and continuous. Uniqueness is clear. ∎

Proposition 4.7. *Let $P \in \mathbf{P}$. Then*

$$\Big((\overline{\Pi})_{\Pi\in\mathcal{E}_P}, (\nu_{\overline{\Pi}_1,\overline{\Pi}_2}\colon \overline{\Pi}_1 \to \overline{\Pi}_2)_{\substack{\Pi_1,\Pi_2\in\mathcal{E}_P\\ \Pi_1\le\Pi_2}}\Big)$$

is an inverse system in $\mathbf{P}$ with inverse limit

$$\Big(\overline{P}, (\nu_{\overline{\Pi}}\colon \overline{P} \to \overline{\Pi})_{\Pi\in\mathcal{E}_P}\Big).$$

Proof. Using Proposition 6, we see that

$$\overline{\mathcal{T}} := \Big((\overline{\Pi})_{\Pi\in\mathcal{E}_P}, (\nu_{\overline{\Pi}_1,\overline{\Pi}_2}\colon \overline{\Pi}_1 \to \overline{\Pi}_2)_{\substack{\Pi_1,\Pi_2\in\mathcal{E}_P\\ \Pi_1\le\Pi_2}}\Big)$$

is an inverse system in $\mathbf{P}$ with which

$$\Big(\overline{P}, (\nu_{\overline{\Pi}}\colon \overline{P} \to \overline{\Pi})_{\Pi\in\mathcal{E}_P}\Big).$$

is compatible.

Assume

$$\Big(Q, (\bar{g}_\Pi\colon Q \to \overline{\Pi})_{\Pi\in\mathcal{E}_P}\Big)$$

is also compatible with $\overline{\mathcal{T}}$. For each $\Pi \in \mathcal{E}_P$, let $g_\Pi\colon Q \to \Pi$ be the continuous order-preserving function $g_\Pi(q) := \bar{g}_\Pi(q)$ $(q \in Q)$. Then

$$\Big(Q, (g_\Pi\colon Q \to \Pi)_{\Pi\in\mathcal{E}_P}\Big)$$

is compatible with the inverse system

$$\Big((\Pi)_{\Pi\in\mathcal{E}_P}, (\nu_{\Pi_1,\Pi_2}\colon \Pi_1 \to \Pi_2)_{\substack{\Pi_1,\Pi_2\in\mathcal{E}_P\\ \Pi_1\le\Pi_2}}\Big)$$

(see Proposition 6). Hence there is a unique continuous order-preserving function $g\colon Q \to P$ such that $\nu_\Pi \circ g = g_\Pi$ for all $\Pi \in \mathcal{E}_P$.

For all q, $r \in Q$, $q \le r$ implies $g(q) = g(r)$. For otherwise by Lemma 5 there exists $\Pi \in \mathcal{E}_P$ such that $\nu_\Pi\Big(g(q)\Big) \not\geq \nu_\Pi\Big(g(r)\Big)$ so that $g_\Pi(q) \not\geq g_\Pi(r)$ and hence $\bar{g}_\Pi(q) \not\geq \bar{g}_\Pi(r)$. As $\overline{\Pi}$ is an antichain, we have $\bar{g}_\Pi(q) \not\leq \bar{g}_\Pi(r)$, so that $\bar{g}_\Pi$ is not order-preserving, a contradiction.

Hence the map $\bar{g}\colon Q \to \overline{P}$ defined by $\bar{g}(q) := g(q)$ for all $q \in Q$ is continuous and order-preserving. Moreover for all $\Pi \in \mathcal{E}_P$, $\nu_{\overline{\Pi}} \circ \bar{g} = \bar{g}_\Pi$.

Assume $\bar{h}\colon Q \to \overline{P}$ is a continuous order-preserving map such that $\nu_{\overline{\Pi}} \circ \bar{h} = \bar{g}_\Pi$ $(\Pi \in \mathcal{E}_P)$. Define $h\colon Q \to P$ by $h(q) := \bar{h}(q)$ $(q \in Q)$. Then h is continuous and order-preserving, and $\nu_\Pi \circ h = g_\Pi$ $(\Pi \in \mathcal{E}_P)$; hence $h = g$, so that $\bar{h} = \bar{g}$. ∎

Lemma 4.8. *Let F be a filtered poset, $\Big(P, (g_i\colon P \to P_i)_{i\in F}\Big)$ an inverse limit in $\mathbf{P}$ of the inverse system*

$$\Big((P_i)_{i\in F}, (g_{ij}\colon P_i \to P_j)_{\substack{i,j\in F\\ i\le j}}\Big).$$

Then:

(1) *$i \le j$ implies $\operatorname{Im} D(g_j) \subseteq \operatorname{Im} D(g_i)$ $(i, j \in F)$;*

(2) *$\{\operatorname{Im} D(g_i) \mid i \in F\}$ is directed;*

(3) *$D(P) = \bigcup_{i\in F} \operatorname{Im} D(g_i)$.*

Proof. (1) Let $i, j \in F$ be such that $i \le j$. Then $g_{ij} \circ g_i = g_j$ implies

$$D(g_i) \circ D(g_{ij}) = D(g_j),$$

so that $\operatorname{Im} D(g_j) \subseteq \operatorname{Im} D(g_i)$.

(2) This statement follows from (1) and the fact F is filtered.

(3) By Priestley duality,

$$\mathcal{S} := \bigg(\Big(D(P_i)\Big)_{i\in F}, \Big(D(g_{ij})\colon D(P_j) \to D(P_i)\Big)_{\substack{i,j\in F\\ i\le j}}\bigg)$$

is a filtered system in $\mathbf{D}$ with filtered limit

$$\bigg(D(P), \Big(D(g_i)\colon D(P_i) \to D(P)\Big)_{i\in F}\bigg).$$

Let $\mathcal{D} := \{\operatorname{Im} D(g_i) \mid i \in F\}$. Then $M := \bigcup \mathcal{D}$ is a $\{0,1\}$-sublattice of $L := D(P)$ by (2). For $i \in F$, let $f_i'\colon D(P_i) \to M$ be the $\{0,1\}$-homomorphism defined by $f_i'(a) := [D(g_i)](a)$ $\Big(a \in D(P_i)\Big)$. For $i, j \in F$ such that $i \le j$ and $a \in D(P_j)$,

$$\begin{aligned}[f_i' \circ D(g_{ij})](a) &= [D(g_i) \circ D(g_{ij})](a) = [D(g_{ij} \circ g_i)](a)\\ &= [D(g_j)](a) = f_j'(a)\end{aligned}$$

so that $\Big(M, (f_i'\colon D(P_i) \to M)_{i\in F}\Big)$ is compatible with $\mathcal{S}$. Hence there exists a unique $\{0,1\}$-homomorphism $f\colon L \to M$ such that $f \circ D(g_i) = f_i'$ $(i \in F)$. For all $i \in F$ and $a \in D(P_i)$, $[f \circ D(g_i)](a) = f_i'(a) = [D(g_i)](a)$.

Let $h\colon L \to L$ be the $\{0,1\}$-homomorphism defined by $h(a) := f(a)$ $(a \in L)$. As $h \circ D(g_i) = D(g_i)$ $(i \in F)$, we see that $h = \mathrm{id}_L$, so that $\operatorname{Im} f = L$ and hence $M = L$. ∎

Lemma 4.9. *Let F and K be filtered posets. Let $\Big(P, (g_i\colon P \to P_i)_{i\in F}\Big)$ be an inverse limit in $\mathbf{P}$ of the inverse system*

$$\Big((P_i)_{i\in F}, (g_{ij}\colon P_i \to P_j)_{\substack{i,j\in F\\ i\le j}}\Big).$$

Let $\Big(P, (h_k\colon P \to Q_k)_{k\in K}\Big)$ be an inverse limit in $\mathbf{P}$ of the inverse system

$$\Big((Q_k)_{k\in K}, (h_{km}\colon Q_k \to Q_m)_{\substack{k,m\in K\\ k\le m}}\Big).$$

Assume that $g_i\colon P \to P_i$ and $h_k\colon P \to Q_k$ are surjective and that P_i and Q_k are finite antichains ($i \in F$, $k \in K$).

Then for all $k \in K$, there exists $i \in F$ for which the following holds: for all $p_i \in P_i$, there exists $q_k \in Q_k$ such that $g_i^{-1}(p_i) \subseteq h_k^{-1}(q_k)$.

Proof. Let $k \in K$. By Lemma 8 (3), $\operatorname{Im} D(h_k) \subseteq \bigcup_{i\in F} \operatorname{Im} D(g_i)$. Hence by Lemma 8 (2) there exists $i \in F$ such that $\operatorname{Im} D(h_k) \subseteq \operatorname{Im} D(g_i)$.

As $\operatorname{Im} D(h_k)$ is a $\{0,1\}$-sublattice of $\operatorname{Im} D(g_i)$,

$$a \le 1_{D(P)} = \bigvee\{\, b \in D(P) \mid b \text{ is an atom of } \operatorname{Im} D(h_k) \,\}$$

for every atom a of $\operatorname{Im} D(g_i)$, so there exists an atom $b \in \operatorname{Im} D(h_k)$ such that $a \le b$. That is, for every $p_i \in P_i$, there exists $q_k \in Q_k$ such that $g_i^{-1}(p_i) \subseteq h_k^{-1}(q_k)$. ■

The next result is easily seen to be true.

Lemma 4.10. *Let P, $Q \in \mathbf{P}$, $M \in \mathbf{Lat} \cup \mathbf{Slat}$. Let $\nu\colon P \to Q$ be a continuous order-preserving map. Define $\mu\colon M^Q \to M^P$ by $\mu(f) := f \circ \nu$ for all $f \in M^Q$. Then:*

(1) *μ is a morphism;*
(2) *μ is injective if ν is surjective.*

Lemma 2 (2) and (3) and Lemma 10 yield the following.

Lemma 4.11. *Let $P \in \mathbf{P}$, $M \in \mathbf{Lat} \cup \mathbf{Slat}$.*

(1) *For every $\Pi \in \mathcal{E}_P$, $\mu_\Pi^M\colon M^\Pi \to M^P$ is an injective morphism;*
(2) *For every Π, $\Pi_0 \in \mathcal{E}_P$ such that $\Pi \le \Pi_0$,*

$$\mu_{\Pi,\Pi_0}^M\colon M^{\Pi_0} \to M^\Pi$$

is an injective morphism.

Lemma 4.12. *Let $P \in \mathbf{P}$, $M \in \mathbf{Lat} \cup \mathbf{Slat}$.*

(1) *For $\Pi \in \mathcal{E}_P$, $\mu^M_{\Pi,\Pi} = \mathrm{id}(M^\Pi)$.*

(2) *For Π_1, Π_2, $\Pi_3 \in \mathcal{E}_P$ such that $\Pi_1 \le \Pi_2 \le \Pi_3$,*

$$\mu^M_{\Pi_1,\Pi_2} \circ \mu^M_{\Pi_2,\Pi_3} = \mu^M_{\Pi_1,\Pi_3}.$$

(3) *For Π_1, $\Pi_2 \in \mathcal{E}_P$ such that $\Pi_1 \le \Pi_2$,*

$$\mu^M_{\Pi_1} \circ \mu^M_{\Pi_1,\Pi_2} = \mu^M_{\Pi_2}.$$

Proof. (1) This part is obvious.

(2) Let $f \in M^{\Pi_3}$. Then

$$\begin{aligned}(\mu^M_{\Pi_1,\Pi_2} \circ \mu^M_{\Pi_2,\Pi_3})(f) &= f \circ \nu_{\Pi_2,\Pi_3} \circ \nu_{\Pi_1,\Pi_2} \\ &= f \circ \nu_{\Pi_1,\Pi_3} = \mu^M_{\Pi_1,\Pi_3}(f).\end{aligned}$$

(3) Let $f \in M^{\Pi_2}$. Then

$$\begin{aligned}(\mu^M_{\Pi_1} \circ \mu^M_{\Pi_1,\Pi_2})(f) &= f \circ \nu_{\Pi_1,\Pi_2} \circ \nu_{\Pi_1} \\ &= f \circ \nu_{\Pi_2} = \mu^M_{\Pi_2}(f)\end{aligned}$$

by Lemma 1. ∎

Lemma 4.13. *Let F be a filtered poset. Let*

$$\mathcal{S} := \Big((C_i)_{i\in F}, (f_{ij}\colon C_j \to C_i)_{\substack{i,j\in F\\ i\le j}}\Big)$$

be a filtered system in $\mathbf{Lat} \cup \mathbf{Slat}$ with which $\Big(C, (f_i\colon C_i \to C)_{i\in F}\Big)$ is compatible. Assume:

(1) *$C = \bigcup_{i\in F} \mathrm{Im}\, f_i$;*

(2) *for all $i \in F$, f_i is injective.*

Then $\Big(C, (f_i\colon C_i \to C)_{i\in F}\Big)$ is a filtered limit of $\mathcal{S}$.

Proof. Assume $\Big(C', (f'_i\colon C_i \to C')_{i\in F}\Big)$ is compatible with $\mathcal{S}$. Define

$$f\colon C \to C'$$

as follows: if $c \in C$ and $c = f_i(c_i)$ for some $i \in F$ and $c_i \in C_i$, let $f(c) := f'_i(c_i) \in C'$.

The map is well-defined. For if $c = f_j(c_j) = f_k(c_k)$ for some j, $k \in F$, $c_j \in C_j$, $c_k \in C_k$, there exists $i \in F$ such that $i \le j$, k. Hence

$$c = (f_i \circ f_{ij})(c_j) = (f_i \circ f_{ik})(c_k),$$

so that $f_{ij}(c_j) = f_{ik}(c_k)$. Now $(f'_i \circ f_{ij})(c_j) = f'_j(c_j)$ and $(f'_i \circ f_{ik})(c_k) = f'_k(c_k)$.

If c, $d \in C$, then there exist j, $k \in F$ such that $c = f_j(c_j)$ and $d = f_k(c_k)$ for some $c_j \in C_j$ and $c_k \in C_k$. There exists $i \in F$ such that $i \le j$, k, and $c = (f_i \circ f_{ij})(c_j)$, $d = (f_i \circ f_{ik})(c_k)$. Thus $c \vee d = f_i\Big(f_{ij}(c_j) \vee f_{ik}(c_k)\Big)$, so

$$\begin{aligned} f(c \vee d) &= f'_i\Big(f_{ij}(c_j) \vee f_{ik}(c_k)\Big) \\ &= (f'_i \circ f_{ij})(c_j) \vee (f'_i \circ f_{ik})(c_k) \\ &= f'_j(c_j) \vee f'_k(c_k) = f(c) \vee f(d). \end{aligned}$$

(If $f \in \mathbf{Lat}$, then it preserves meet as well.) Hence f is a morphism. Uniqueness is clear. ∎

Cf. (2) below with [18], Theorem V.4.1.

Proposition 4.14. *Let $P \in \mathbf{P}$, $M \in \mathbf{Lat} \cup \mathbf{Slat}$. Then:*

(1) $M^P = \bigcup_{\Pi \in \mathcal{E}_P} \operatorname{Im} \mu^M_\Pi$;

(2) $\Big(M^P, (\mu^M_\Pi\colon M^\Pi \to M^P)_{\Pi \in \mathcal{E}_P}\Big)$ *is a filtered limit of the filtered system*

$$\Big((M^\Pi)_{\Pi \in \mathcal{E}_P}, (\mu^M_{\Pi_1,\Pi_2}\colon M^{\Pi_2} \to M^{\Pi_1})_{\substack{\Pi_1,\Pi_2 \in \mathcal{E}_P \\ \Pi_1 \le \Pi_2}}\Big);$$

(3) *for* Π_1, $\Pi_2 \in \mathcal{E}_P$, $\Pi_1 \le \Pi_2$ *implies*

$$\operatorname{Im} \mu^M_{\Pi_2} \subseteq \operatorname{Im} \mu^M_{\Pi_1}.$$

Proof. (1) Let $f \in M^P$. By Lemma 3, there exist $\Pi \in \mathcal{E}_P$ and an order-preserving map $g\colon \Pi \to M$ such that $f = g \circ \nu_\Pi = \mu_\Pi^M(g)$.

(2) By Proposition 4, $\mathcal{E}_P$ is filtered. By Lemma 12,

$$\mathcal{S} := \Big((M^\Pi)_{\Pi \in \mathcal{E}_P}, (\mu^M_{\Pi_1,\Pi_2}\colon M^{\Pi_2} \to M^{\Pi_1})_{\substack{\Pi_1,\Pi_2 \in \mathcal{E}_P \\ \Pi_1 \le \Pi_2}} \Big)$$

is a filtered system with which

$$\Big(M^P, (\mu^M_\Pi \colon M^\Pi \to M^P)_{\Pi \in \mathcal{E}_P} \Big)$$

is compatible. By (1) and Lemmas 11 and 13, it is a filtered limit of $\mathcal{S}$.

(3) This part follows from (2). ■

Proposition 4.15. *Let $P \in \mathbf{P}$, $M \in \mathbf{Lat} \cup \mathbf{Slat}$. Then:*

(1) $M^{\overline{P}} = \bigcup_{\Pi \in \mathcal{E}_P} \operatorname{Im} \mu^M_{\overline{\Pi}}$;

(2) $\Big(M^{\overline{P}}, (\mu^M_{\overline{\Pi}} \colon M^{\overline{\Pi}} \to M^{\overline{P}})_{\Pi \in \mathcal{E}_P} \Big)$ *is a filtered limit of the filtered system*

$$\Big((M^{\overline{\Pi}})_{\Pi \in \mathcal{E}_P}, (\mu^M_{\overline{\Pi}_1,\overline{\Pi}_2}\colon M^{\overline{\Pi}_2} \to M^{\overline{\Pi}_1})_{\substack{\Pi_1,\Pi_2 \in \mathcal{E}_P \\ \Pi_1 \le \Pi_2}} \Big);$$

(3) *for* Π_1, $\Pi_2 \in \mathcal{E}_P$, $\Pi_1 \le \Pi_2$ *implies*

$$\operatorname{Im} \mu^M_{\overline{\Pi}_2} \subseteq \operatorname{Im} \mu^M_{\overline{\Pi}_1}.$$

Proof. (1) Let $f \in M^{\overline{P}}$. By Lemma 3, there exist $\Pi_0 \in \mathcal{E}_{\overline{P}}$ and a map $g \in M^{\Pi_0}$ such that

$$f = g \circ \nu_{\Pi_0}.$$

By Propositions 6 and 7 and Lemma 9, there exists $\Pi \in \mathcal{E}_P$ such that $\Pi \le \Pi_0$. By Lemma 1,

$$f = g \circ \nu_{\Pi_0} = g \circ \nu_{\overline{\Pi},\Pi_0} \circ \nu_{\overline{\Pi}} \in \operatorname{Im} \mu^M_{\overline{\Pi}}.$$

(2) By Proposition 14,

$$\overline{\mathcal{S}} := \Big((M^{\overline{\Pi}})_{\Pi \in \mathcal{E}_P}, (\mu^M_{\overline{\Pi}_1,\overline{\Pi}_2}\colon M^{\overline{\Pi}_2} \to M^{\overline{\Pi}_1})_{\substack{\Pi_1,\Pi_2 \in \mathcal{E}_P \\ \Pi_1 \le \Pi_2}} \Big)$$

is a filtered system with which

$$\Big(M^{\overline{P}}, (\mu^M_{\overline{\Pi}} \colon M^{\overline{\Pi}} \to M^{\overline{P}})_{\Pi \in \mathcal{E}_P} \Big)$$

is compatible.

By (1) and Lemmas 11 and 13, it is a filtered limit of $\overline{\mathcal{S}}$.

(3) This part follows from (2). ■

5. Compact congruences of Priestley powers

In this section we prove that, for $L \in \mathbf{Lat}$ and $P \in \mathbf{P}$, $\operatorname{Comp}(L^P) \cong (\operatorname{Comp} L)^{\overline{P}}$ (Theorem 10). We use a suggestion from [23], pp. 98–100 about obtaining the semilattice of compact congruences of a limit of lattices L_i as a limit of the semilattices $\operatorname{Comp} L_i$.

In [23], p. 98 and [11], §3, two prescriptions are given for functors from **Lat** to **Slat** given by $L \mapsto \operatorname{Comp} L$ ($L \in \mathbf{Lat}$). We show that these two prescriptions yield the same functor (Lemma 3).

Our first lemma is a consequence of [19], Theorem 1.20. The second is a corollary, but we use an easy proof suggested by Dr. P. M. Neumann.

Lemma 5.1. *Let $L \in \mathbf{Lat}$, $X \subseteq L \times L$. Define $A_n(X)$ ($n \geq 0$) by induction:*

$$A_0(X) := X \cup \{\, (a,b) \mid (b,a) \in X \,\} \cup \{\, (a,a) \mid a \in L \,\};$$
$$A_{n+1}(X) := A_n(X) \cup Q_n(X) \cup T_n(X)$$

where

$$Q_n(X) := \{\, (a_1 \vee a_2, b_1 \vee b_2), (a_1 \wedge a_2, b_1 \wedge b_2) \mid (a_i, b_i) \in A_n(X)\ (i = 1,\ 2) \,\},$$
$$T_n(X) := \{\, (a,c) \mid (a,b),\ (b,c) \in A_n(X) \text{ for some } b \in L \,\}.$$

Then $\vartheta^L(X) = \bigcup_{n \geq 0} A_n(X)$.

Lemma 5.2. *Let L, $M \in \mathbf{Lat}$, $X \subseteq L \times L$. Let $f\colon L \to M$ be a homomorphism. Then*

$$(f \times f)[\vartheta^L(X)] \subseteq \vartheta^M\Big((f \times f)[X]\Big).$$

Proof. Let $\rho := \vartheta^M\Big((f \times f)[X]\Big)$, and let $\varsigma := (f \times f)^{-1}(\rho)$. Since f is a homomorphism, $\varsigma \in \operatorname{Con} L$. Clearly $X \subseteq \varsigma$, so $\vartheta^L(X) \subseteq \varsigma$. Hence

$$(f \times f)[\vartheta^L(X)] \subseteq \vartheta^M\Big((f \times f)[X]\Big).$$

■

Lemma 5.3. *Let L, $M \in \mathbf{Lat}$, and let $f: L \to M$ be a homomorphism. Let $n \geq 0$, a_i, $b_i \in L$ ($i = 1,\ldots, n$). Then*

$$[\mathrm{Comp}(f)]\Big(\bigvee_{i=1}^{n} \vartheta^L(a_i, b_i)\Big) = \bigvee_{i=1}^{n} \vartheta^M\Big(f(a_i), f(b_i)\Big).$$

Hence Comp *is a functor from* **Lat** *to* **Slat**.

Proof. Clearly $\bigvee_{i=1}^{n} \vartheta^M\Big(f(a_i), f(b_i)\Big) \subseteq [\mathrm{Comp}(f)]\Big(\bigvee_{i=1}^{n} \vartheta^L(a_i, b_i)\Big)$. By Lemma 2,

$$(f \times f)[\bigvee_{i=1}^{n} \vartheta^L(a_i, b_i)] \subseteq \bigvee_{i=1}^{n} \vartheta^M\Big(f(a_i), f(b_i)\Big),$$

so that

$$[\mathrm{Comp}(f)]\Big(\bigvee_{i=1}^{n} \vartheta^L(a_i, b_i)\Big) \subseteq \bigvee_{i=1}^{n} \vartheta^M\Big(f(a_i), f(b_i)\Big).$$

Thus

$$[\mathrm{Comp}(f)]\Big(\bigvee_{i=1}^{n} \vartheta^L(a_i, b_i)\Big) = \bigvee_{i=1}^{n} \vartheta^M\Big(f(a_i), f(b_i)\Big).$$

■

In [9], Theorem 2.1, it is proven that, for $L \in \mathbf{Lat}$, $P \in \mathbf{P}^{\mathrm{fin}}$, $\mathrm{Con}(L^P) \cong (\mathrm{Con}\, L)^n$, where n is the cardinality of P. (Also see a similar result for certain lattice-ordered algebras, [8], Theorem 3.5.) The proof is by induction on n. We present essentially the same proof below, only we have made it direct.

First we state some lemmas.

Lemma 5.4. *Let $L \in \mathbf{Lat}$, $P \in \mathbf{P}^{\mathrm{fin}}$, a, $b \in L$, $p \in P$. Then*

$$m_P(a, b, p) \in L^P$$

■

Lemma 5.5. *Let A, B be algebraic lattices. Then $\kappa(A \times B) = \kappa(A) \times \kappa(B)$. For $n \geq 0$, $\kappa(A^n) = \kappa(A)^n$.*

Lemma 5.6. *Let $L \in \mathbf{Lat}$, $P \in \mathbf{P}^{\mathrm{fin}}$, $p \in P$, a, $b \in L$, f_0, $g_0 \in L^P$, $\theta \in \mathrm{Con}(L^P)$. Assume $f_0(p) = a$, $g_0(p) = b$, and $(f_0, g_0) \in \theta \vee \chi_p$. Then*

$$(f, g) \in \theta \vee \chi_p$$

for all f, $g \in L^P$ such that $f(p) = a$, $g(p) = b$.

Proposition 5.7. *Let $L \in \mathbf{Lat}$, $P \in \mathbf{P}^{\mathrm{fin}}$. Then:*
(1) *for $\theta \in \operatorname{Con}(L^P)$ and $p \in P$,*

$$[\Gamma'_P(\theta)](p) = \{\, (a,b) \in L \times L \mid (f,g) \in \theta \vee \chi_p \text{ for some } f,\ g \in L^P \text{ such that } f(p) = a,\ g(p) = b \,\};$$

(2) *Γ'_P and Δ'_P are mutually-inverse order-isomorphisms;*
(3) *Γ_P maps $\operatorname{Comp}(L^P)$ onto $(\operatorname{Comp} L)^{\overline{P}}$.*

Proof. For $a \in L$, let $\bar{a} \in L^P$ denote the constant map with value a.

By Lemma 6, $\Gamma' := \Gamma'_P$ is well-defined, as is $\Delta' := \Delta'_P$; (1) also holds. Both Γ' and Δ' are order-preserving.

Let $\theta \in \operatorname{Con}(L^P)$. Let f, $g \in L^P$. Then

$$\begin{aligned}(f,g) \in (\Delta' \circ \Gamma')(\theta) &\iff \bigl(f(p), g(p)\bigr) \in [\Gamma'(\theta)](p) \text{ for all } p \in P\\ &\iff (f,g) \in \theta \vee \chi_p \text{ for all } p \in P\\ &\iff (f,g) \in \bigwedge_{p\in P} (\theta \vee \chi_p)\\ &\iff (f,g) \in \theta \vee \bigwedge_{p\in P} \chi_p\\ &\iff (f,g) \in \theta.\end{aligned}$$

Thus $(\Delta' \circ \Gamma')(\theta) = \theta$, so that $\Delta' \circ \Gamma' = \mathrm{id}_{\operatorname{Con}(L^P)}$.

Let $F \in (\operatorname{Con} L)^{\overline{P}}$, a, $b \in L$, $p_0 \in P$. First assume $(a,b) \in F(p_0)$. Then for all $p \in P$, $\bigl([m_P(a,b,p_0)](p), [m_P(b,a,p_0)](p)\bigr) \in F(p)$, so that

$$\bigl(m_P(a,b,p_0), m_P(b,a,p_0)\bigr) \in \Delta'(F)$$

and hence

$$(a,b) \in [(\Gamma' \circ \Delta')(F)](p_0).$$

Therefore $F \leq (\Gamma' \circ \Delta')(F)$.

Now assume $(a,b) \in [(\Gamma' \circ \Delta')(F)](p_0)$. Then

$$(f,g) \in \Delta'(F) \vee \chi_{p_0}$$

for all f, $g \in L^P$ such that $f(p_0) = a$, $g(p_0) = b$. Hence

$$(\bar{a}, \bar{b}) \in \Delta'(F) \vee \chi_{p_0}.$$

Thus for some $n \geq 1$, there exist $f_1,\ldots,f_n \in L^P$ such that $\bar{a} = f_1$, $\bar{b} = f_n$, and for $1 \leq i \leq n$,

$$(f_i, f_{i+1}) \in \begin{cases} \Delta'(F) & \text{if } i \text{ odd,} \\ \chi_{p_0} & \text{if } i \text{ even.} \end{cases}$$

Therefore $(a, b) \in F(p_0)$. Hence $(\Gamma' \circ \Delta')(F) \leq F$, so that $(\Gamma' \circ \Delta')(F) = F$. We see that $\Gamma' \circ \Delta' = \mathrm{id}_{(\mathrm{Con}\, L)^{\overline{P}}}$. Thus Γ' and Δ' are inverse order-isomorphisms, which is (2).

Statement (3) follows from (2) and Lemma 5. ∎

Lemma 5.8. *Let $L \in \mathbf{Lat}$, P, $Q \in \mathbf{P}^{\mathrm{fin}}$. Let $\nu: P \to Q$ be order-preserving. Let $\bar{\nu}: \overline{P} \to \overline{Q}$ be defined by $\bar{\nu}(p) := \nu(p)$ for all $p \in P$. Define $\mu: L^Q \to L^P$ by*

$$\mu(f) := f \circ \nu \qquad (f \in L^Q).$$

Define $\bar{\mu}: (\mathrm{Comp}\, L)^{\overline{Q}} \to (\mathrm{Comp}\, L)^{\overline{P}}$ by

$$\bar{\mu}(F) := F \circ \bar{\nu} \qquad (F \in (\mathrm{Comp}\, L)^{\overline{Q}}).$$

Then:

(1)
$$\Gamma_P \circ \mathrm{Comp}(\mu) \circ \Delta_Q: (\mathrm{Comp}\, L)^{\overline{Q}} \to (\mathrm{Comp}\, L)^{\overline{P}}$$
is a $\{0\}$-$\vee$-homomorphism and equals $\bar{\mu}$;

(2) *if ν is surjective, then* $\mathrm{Comp}\,\mu$ *is injective.*

Proof. By Lemma 4.10, μ and $\bar{\mu}$ are **Lat**- and **Slat**-morphisms, respectively, so

$$\mathrm{Comp}(\mu): \mathrm{Comp}(L^Q) \to \mathrm{Comp}(L^P)$$

is defined. By Proposition 7, $\Gamma_P \circ \mathrm{Comp}(\mu) \circ \Delta_Q$ is an **Slat**-morphism. Fix a_0, $b_0 \in L$, $q_0 \in Q$. To prove (1), it suffices to show

$$\Big(\Gamma_P \circ \mathrm{Comp}(\mu) \circ \Delta_Q\Big)(F) = \bar{\mu}(F)$$

for $F \in (\mathrm{Comp}\, L)^{\overline{Q}}$ defined by

$$F(q) = \begin{cases} \vartheta^L(a_0, b_0) & \text{if } q = q_0, \\ 0_{\mathrm{Con}\, L} & \text{if } q \neq q_0. \end{cases}$$

Thus for f, $g \in L^Q$, $(f, g) \in \Delta_Q(F)$ if and only if

(1) $\big(f(q_0), g(q_0)\big) \in \vartheta^L(a_0, b_0)$ and
(2) $f(q) = g(q)$ for all $q \in Q \setminus \{q_0\}$.

Let

$$\eta := \{ (h,k) \in L^P \times L^P \mid \big(h(p), k(p)\big) \in \vartheta^L(a_0, b_0) \text{ for all } p \in \nu^{-1}(q_0) \\ \text{and } h(p) = k(p) \text{ for all } p \in P \setminus \nu^{-1}(q_0) \}.$$

Then $\eta \in \mathrm{Con}(L^P)$ by Proposition 7, and $(\mu \times \mu)[\Delta_Q(F)] \subseteq \eta$.

Assume $\theta \in \mathrm{Con}(L^P)$ and $(\mu \times \mu)[\Delta_Q(F)] \subseteq \theta$. Assume $(a, b) \in \vartheta^L(a_0, b_0)$. Then $\big(m_Q(a, b, q_0), m_Q(b, a, q_0)\big) \in \Delta_Q(F)$; therefore

$$(\mu \times \mu)\Big(m_Q(a, b, q_0), m_Q(b, a, q_0)\Big) \in \theta,$$

and so $\Big(m_Q(a, b, q_0) \circ \nu, m_Q(b, a, q_0) \circ \nu\Big) \in \theta$, thus $(a, b) \in [\Gamma'_P(\theta)](p)$ for all $p \in \nu^{-1}(q_0)$. Hence

$$\vartheta^L(a_0, b_0) \subseteq [\Gamma'_P(\theta)](p)$$

for all $p \in \nu^{-1}(q_0)$. If $(h, k) \in \eta$ then

$$\Big(h(p), k(p)\Big) \in [\Gamma'_P(\theta)](p)$$

for all $p \in P$, so $(h, k) \in \theta$. Hence $\eta \subseteq \theta$. We have shown that

$$\eta = (\mathrm{Comp}\,\mu)\Big(\Delta_Q(F)\Big).$$

Define $G \in (\mathrm{Comp}\,L)^{\overline{P}}$ as follows: for all $p \in P$,

$$G(p) := \begin{cases} \vartheta^L(a_0, b_0) & \text{if } p \in \nu^{-1}(q_0), \\ 0_{\mathrm{Con}\,L} & \text{else.} \end{cases}$$

It is clear that $\Delta_P(G) = \eta$. Moreover $G = \bar{\mu}(F)$. Thus (1) holds.

Statement (2) follows from (1) and Lemma 4.10 (2). ■

In [23], p.98 it is stated that **Lat**-embeddings map to **Slat**-embeddings under Comp. One may construct a counterexample by considering the five-element non-distributive modular lattice. The statement does hold for the embeddings with which we are concerned, however.

Lemma 5.9. *Let $L \in \mathbf{Lat}$, $P \in \mathbf{P}$. For all $\Pi \in \mathcal{E}_P$,* $\operatorname{Comp}\mu^L_\Pi$ *is injective.*

Proof. Let $\theta^{(1)}$, $\theta^{(2)} \in \operatorname{Comp}(L^\Pi)$ be such that

$$(\operatorname{Comp}\mu^L_\Pi)(\theta^{(1)}) = (\operatorname{Comp}\mu^L_\Pi)(\theta^{(2)}).$$

For some $n \geq 0$,

$$\theta^{(r)} = \bigvee_{i=1}^{n} \vartheta^{L^\Pi}(f_i^{(r)}, g_i^{(r)})$$

for some $f_i^{(r)}$, $g_i^{(r)} \in L^\Pi$ ($i = 1,\ldots,n$ and $r = 1, 2$). By Lemma 3,

$$(\operatorname{Comp}\mu^L_\Pi)(\theta^{(r)}) = \bigvee_{i=1}^{n} \vartheta^{L^P}\Big(\mu^L_\Pi(f_i^{(r)}), \mu^L_\Pi(g_i^{(r)})\Big) \qquad (r = 1, 2).$$

By Lemma 1, there exists a finite subset $S \subseteq L^P$ containing $\mu^L_\Pi(f_i^{(r)})$, $\mu^L_\Pi(g_i^{(r)})$ ($i = 1,\ldots,n$ and $r = 1, 2$) such that if K is a sublattice of L^P containing S, then

$$\bigvee_{i=1}^{n} \vartheta^{K}\Big(\mu^L_\Pi(f_i^{(1)}), \mu^L_\Pi(g_i^{(1)})\Big) = \bigvee_{i=1}^{n} \vartheta^{K}\Big(\mu^L_\Pi(f_i^{(2)}), \mu^L_\Pi(g_i^{(2)})\Big).$$

As S is finite, by Propositions 4.4 and 4.14 there exists $\Pi' \in \mathcal{E}_P$ such that $\Pi' \leq \Pi$ and $S \subseteq \operatorname{Im}\mu^L_{\Pi'}$. As $\mu^L_\Pi = \mu^L_{\Pi'} \circ \mu^L_{\Pi',\Pi}$ and $\mu^L_{\Pi'}$ is injective [Lemmas 4.11 (1) and 4.12 (3)],

$$\bigvee_{i=1}^{n} \vartheta^{L^{\Pi'}}\Big(\mu^L_{\Pi',\Pi}(f_i^{(1)}), \mu^L_{\Pi',\Pi}(g_i^{(1)})\Big) = \bigvee_{i=1}^{n} \vartheta^{L^{\Pi'}}\Big(\mu^L_{\Pi',\Pi}(f_i^{(2)}), \mu^L_{\Pi',\Pi}(g_i^{(2)})\Big).$$

Hence $(\operatorname{Comp}\mu^L_{\Pi',\Pi})(\theta^{(1)}) = (\operatorname{Comp}\mu^L_{\Pi',\Pi})(\theta^{(2)})$. By Lemma 8, $\operatorname{Comp}\mu^L_{\Pi',\Pi}$ is injective, so $\theta^{(1)} = \theta^{(2)}$.

■

The following proof utilizes an idea from [23], pp. 98–100. The theorem, proved independently, appears in [15], Theorem 4.

Theorem 5.10. *Let $L \in \mathbf{Lat}$, $P \in \mathbf{P}$. Then* $\operatorname{Comp}(L^P) \cong (\operatorname{Comp} L)^{\overline{P}}$.

Proof. By Proposition 4.14 (2),

$$\Big(L^P, (\mu^L_\Pi\colon L^\Pi \to L^P)_{\Pi\in\mathcal{E}_P}\Big)$$

is a filtered limit in **Lat** of the filtered system

$$\Big((L^\Pi)_{\Pi\in\mathcal{E}_P}, (\mu^L_{\Pi_1,\Pi_2}\colon L^{\Pi_2} \to L^{\Pi_1})_{\substack{\Pi_1,\Pi_2\in\mathcal{E}_P\\ \Pi_1\le\Pi_2}}\Big).$$

Hence

$$\left(\Big(\mathrm{Comp}(L^\Pi)\Big)_{\Pi\in\mathcal{E}_P}, \Big(\mathrm{Comp}\,\mu^L_{\Pi_1,\Pi_2}\colon \mathrm{Comp}(L^{\Pi_2}) \to \mathrm{Comp}(L^{\Pi_1})\Big)_{\substack{\Pi_1,\Pi_2\in\mathcal{E}_P\\ \Pi_1\le\Pi_2}}\right)$$

is a filtered system in **Slat**. By Lemma 8 (1), for Π_1, $\Pi_2\in\mathcal{E}_P$ such that $\Pi_1\le\Pi_2$,

$$\Gamma_{\Pi_1}\circ \mathrm{Comp}(\mu^L_{\Pi_1,\Pi_2})\circ\Delta_{\Pi_2} = \mu^{\mathrm{Comp}\,L}_{\overline{\Pi}_1,\overline{\Pi}_2}\colon (\mathrm{Comp}\,L)^{\overline{\Pi}_2} \to (\mathrm{Comp}\,L)^{\overline{\Pi}_1}.$$

By Proposition 4.15,

$$\left(\Big((\mathrm{Comp}\,L)^{\overline{\Pi}}\Big)_{\Pi\in\mathcal{E}_P}, \Big(\mu^{\mathrm{Comp}\,L}_{\overline{\Pi}_1,\overline{\Pi}_2}\colon (\mathrm{Comp}\,L)^{\overline{\Pi}_2} \to (\mathrm{Comp}\,L)^{\overline{\Pi}_1}\Big)_{\substack{\Pi_1,\Pi_2\in\mathcal{E}_P\\ \Pi_1\le\Pi_2}}\right)$$

is a filtered system in **Slat** with filtered limit

$$\Big((\mathrm{Comp}\,L)^{\overline{P}}, (\mu^{\mathrm{Comp}\,L}_{\overline{\Pi}}\colon (\mathrm{Comp}\,L)^{\overline{\Pi}} \to (\mathrm{Comp}\,L)^{\overline{P}})_{\Pi\in\mathcal{E}_P}\Big).$$

For each $\Pi\in\mathcal{E}_P$, let $f'_\Pi := \mathrm{Comp}(\mu^L_\Pi)\circ\Delta_\Pi\colon (\mathrm{Comp}\,L)^{\overline{\Pi}} \to \mathrm{Comp}(L^P)$. For Π_1, $\Pi_2\in\mathcal{E}_P$ such that $\Pi_1\le\Pi_2$,

$$\begin{aligned} f'_{\Pi_1}\circ\mu^{\mathrm{Comp}\,L}_{\overline{\Pi}_1,\overline{\Pi}_2} &= \mathrm{Comp}(\mu^L_{\Pi_1})\circ\Delta_{\Pi_1}\circ\Gamma_{\Pi_1}\circ\mathrm{Comp}(\mu^L_{\Pi_1,\Pi_2})\circ\Delta_{\Pi_2}\\ &= \mathrm{Comp}(\mu^L_{\Pi_1})\circ\mathrm{Comp}(\mu^L_{\Pi_1,\Pi_2})\circ\Delta_{\Pi_2}\\ &= \mathrm{Comp}(\mu^L_{\Pi_1}\circ\mu^L_{\Pi_1,\Pi_2})\circ\Delta_{\Pi_2}\\ &= \mathrm{Comp}(\mu^L_{\Pi_2})\circ\Delta_{\Pi_2}\\ &= f'_{\Pi_2}\end{aligned}$$

by Lemma 4.12 (3). Hence there exists a unique **Slat**-morphism

$$F\colon (\mathrm{Comp}\,L)^{\overline{P}} \to \mathrm{Comp}(L^P)$$

such that

$$F \circ \mu_{\overline{\Pi}}^{\operatorname{Comp} L} = \operatorname{Comp}(\mu_\Pi^L) \circ \Delta_\Pi$$

for all $\Pi \in \mathcal{E}_P$. By Lemma 4.11, $\mu_{\overline{\Pi}}^{\operatorname{Comp} L}$ is injective for all $\Pi \in \mathcal{E}_P$. If f_1, $f_2 \in (\operatorname{Comp} L)^{\overline{P}}$ and $F(f_1) = F(f_2)$, then by Proposition 4.15 there exist $\Pi \in \mathcal{E}_P$ and g_1, $g_2 \in (\operatorname{Comp} L)^{\overline{\Pi}}$ such that

$$f_i = \mu_{\overline{\Pi}}^{\operatorname{Comp} L}(g_i) \qquad (i = 1,\, 2).$$

Hence $\Big(\operatorname{Comp}(\mu_\Pi^L) \circ \Delta_\Pi\Big)(g_1) = \Big(\operatorname{Comp}(\mu_\Pi^L) \circ \Delta_\Pi\Big)(g_2)$. By Proposition 7 and Lemma 9, $g_1 = g_2$, so that $f_1 = f_2$. Therefore F is injective.

Now assume $\theta \in \operatorname{Comp}(L^P)$. Then for some $n \geq 0$, there exist $f_1, \ldots, f_n$, $g_1, \ldots, g_n \in L^P$ such that

$$\theta = \bigvee_{i=1}^{n} \vartheta^{L^P}(f_i, g_i).$$

By Proposition 4.14, there exists $\Pi \in \mathcal{E}_P$ such that f_i, $g_i \in \operatorname{Im} \mu_\Pi^L$ $(i = 1, \ldots, n)$. Let h_i, $k_i \in L^\Pi$ be such that $f_i = \mu_\Pi^L(h_i)$, $g_i = \mu_\Pi^L(k_i)$ $(i = 1, \ldots, n)$. Then

$$\bigvee_{i=1}^{n} \vartheta^{L^\Pi}(h_i, k_i) \in \operatorname{Comp}(L^\Pi)$$

and by Lemma 3

$$(\operatorname{Comp} \mu_\Pi^L)\Big(\bigvee_{i=1}^{n} \vartheta^{L^\Pi}(h_i, k_i)\Big) = \bigvee_{i=1}^{n} \vartheta^{L^P}(f_i, g_i) = \theta$$

so that F is surjective. Hence F is an isomorphism. ■

6. The congruence lattice of a Priestley power of a lattice

In this section we determine the structure of the congruence lattice of a Priestley power of a lattice in terms of the lattice and the Priestley space (Theorem 7 and Corollaries 8, 10, and 11). We derive as corollaries the known results that, when the lattice or the space is finite, the problem of §1 has a positive solution (Corollaries 12 and 13).

In §5 we determined the structure of the distributive semilattice of compact congruences of a Priestley power of a lattice. To go from this semilattice to the congruence lattice, we use Stone duality.

Lemma 6.1. *Every trivially ordered Priestley space is a Stone space.*

Proof. Consider a trivially ordered Priestley space. It is homeomorphic to $P(B)$ for some Boolean algebra B. This space has a basis consisting of the sets

$$\{\, F \in P(B) \mid a \in F \,\} \qquad (a \in B).$$

The map

$$F \mapsto B \setminus F \qquad [F \in P(B)]$$

is a bijection from $P(B)$ to $\mathcal{S}(B)$, which has a basis consisting of the sets

$$\{\, I \in B^{\sigma} \mid I \text{ prime and } a \notin I \,\} \qquad (a \in B),$$

so that the map is a homeomorphism. ■

Lemma 6.2 ([29], Lemma 6)**.** *The product of sober spaces is sober.*

Lemma 6.3. *Let X and Y be Stone spaces. Then $X \times Y$ is a Stone space with basis $\{\, U \times V \mid U \in \mathcal{CO}(X),\, V \in \mathcal{CO}(Y) \,\} \subseteq \mathcal{CO}(X \times Y)$.*

Proof. Obviously $X \times Y$ is T_0 and has basis

$$\{\, U \times V \mid U \in \mathcal{CO}(X),\, V \in \mathcal{CO}(Y) \,\} \subseteq \mathcal{CO}(X \times Y).$$

By Lemma 2, it is sober. ■

Lemma 6.4. *Let X and Y be Stone spaces and J a set. For all $j \in J$, let $S_j \in \mathcal{CO}(X)$ and $T_j \in \mathcal{CO}(Y)$. Let $R := \bigcup_{j \in J} (S_j \times T_j) \in \mathcal{O}(X \times Y)$. Then:*

(1) *for all $y \in Y$,*

$$\bigcup\{\, U \in \mathcal{CO}(X) \mid U \times \{y\} \subseteq R \,\} = \bigcup\{\, S_j \mid j \in J \text{ and } y \in T_j \,\};$$

(2) *for all $W \in \mathcal{CO}(X)$,*

$$\left\{ y \in Y \;\middle|\; W \subseteq \bigcup\left\{ U \in \mathcal{CO}(X) \;\middle|\; U \times \{y\} \subseteq R \right\} \right\} \in \mathcal{O}(Y);$$

(3) *for all $y_0 \in Y$,*

$$\bigcap\{\, T_j \mid j \in J \text{ and } y_0 \in T_j \,\} \cap \bigcap\{\, Y \setminus T_j \mid j \in J \text{ and } y_0 \notin T_j \,\}$$

is a subset of the set of all $y \in Y$ such that

$$\bigcup\left\{ U_1 \in \mathcal{CO}(X) \;\middle|\; U_1 \times \{y_0\} \subseteq R \right\} = \bigcup\left\{ U_2 \in \mathcal{CO}(X) \;\middle|\; U_2 \times \{y\} \subseteq R \right\}.$$

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Proof. (1) Fix $y \in Y$. Let $U \in \mathcal{CO}(X)$ be such that $U \times \{y\} \subseteq R$. Then for all $u \in U$ there exists $j_u \in J$ such that

$$(u, y) \in S_{j_u} \times T_{j_u}.$$

Hence $u \in \{ S_j \mid j \in J \text{ and } y \in T_j \}$.

Now assume $j \in J$, $s \in S_j$, and $y \in T_j$. Then $S_j \in \mathcal{CO}(X)$ is such that $S_j \times \{y\} \subseteq R$.

(2) Let $W \in \mathcal{CO}(X)$ and $y_0 \in Y$ be such that

$$W \subseteq \bigcup \{ U \in \mathcal{CO}(X) \mid U \times \{y_0\} \subseteq R \}.$$

By (1), for some $n \geq 0$ there exist $j_1, \dots, j_n \in J$ such that

$$W \subseteq \bigcup_{k=1}^{n} S_{j_k} \qquad \text{and} \qquad y_0 \in \bigcap_{k=1}^{n} T_{j_k} =: T.$$

For any $t \in T$,

$$W \subseteq \bigcup_{k=1}^{n} S_{j_k} \subseteq \bigcup \Big\{ U \in \mathcal{CO}(X) \Bigm| U \times \{t\} \subseteq R \Big\}.$$

As $T \in \mathcal{O}(Y)$, (2) follows.

(3) Assume

$$y \in \bigcap \{ T_j \mid j \in J \text{ and } y_0 \in T_j \} \cap \bigcap \{ Y \setminus T_j \mid j \in J \text{ and } y_0 \notin T_j \}.$$

By (1),

$$\bigcup \Big\{ U_1 \in \mathcal{CO}(X) \Bigm| U_1 \times \{y_0\} \subseteq R \Big\}$$

equals

$$\begin{aligned}\bigcup \{ S_j \mid j \in J \text{ and } y_0 \in T_j \} &= \bigcup \{ S_j \mid j \in J \text{ and } y \in T_j \} \\ &= \bigcup \Big\{ U_2 \in \mathcal{CO}(X) \Bigm| U_2 \times \{y\} \subseteq R \Big\}.\end{aligned}$$

■

The next lemma is simple.

Lemma 6.5. *Let X be a Stone space. Then:*
(1) *$\mathcal{O}(X)$ is an algebraic lattice;*
(2) *$\mathcal{CO}(X) = \kappa\mathcal{O}(X)$.*

After proving the next proposition, the author noted that the first part follows from [12], Theorem II.4.10.

Proposition 6.6. *Let X be a Stone space and Y a trivially ordered Priestley space. Define a map*

$$\Psi\colon \mathcal{O}(X \times Y) \to \mathcal{O}(X)^Y_\Sigma$$

as follows: for all $R \in \mathcal{O}(X \times Y)$ and $y \in Y$, let

$$[\Psi(R)](y) := \bigcup\{\, U \in \mathcal{CO}(X) \mid U \times \{y\} \subseteq R \,\}.$$

Then Ψ is an order-isomorphism. The restriction of Ψ to $\mathcal{CO}(X \times Y)$ maps onto $\mathcal{CO}(X)^Y$.

Proof. By Lemmas 1 and 3, every $R \in \mathcal{O}(X \times Y)$ equals $\bigcup_{j\in J}(S_j \times T_j)$ for some set J and $S_j \in \mathcal{CO}(X)$, $T_j \in \mathcal{CO}(Y)$ $(j \in J)$. (If $R \in \mathcal{CO}(X \times Y)$, we may assume J is finite, so that, by Lemma 4 (1) and (3), $\Psi(R) \in \mathcal{CO}(X)^Y$.) By Lemma 4 (2), Ψ is well-defined. It is clearly order-preserving.

Assume R, $S \in \mathcal{O}(X\times Y)$ and $\Psi(R) \le \Psi(S)$. Assume $(x, y) \in R$. Then there exists $U \in \mathcal{CO}(X)$ such that $x \in U$ and $U \times \{y\} \subseteq R$. Hence

$$U \subseteq [\Psi(R)](y) \subseteq [\Psi(S)](y).$$

Therefore there exists $U_0 \in \mathcal{CO}(X)$ such that $x \in U_0$ and $U_0 \times \{y\} \subseteq S$. Hence $(x, y) \in S$. Therefore $R \subseteq S$ and Ψ is an order-embedding.

Now assume $f \in \mathcal{O}(X)^Y_\Sigma$. Suppose $U \in \mathcal{CO}(X)$, $y \in Y$, and $U \subseteq f(y)$. Then there exists $T_{U,y} \in \mathcal{CO}(Y)$ such that $y \in T_{U,y}$ and $U \subseteq f(t)$ for all $t \in T_{U,y}$. [If $f \in \mathcal{CO}(X)^Y$, let $T_{U,y} := f^{-1}\Big(f(y)\Big)$.]

Let

$$R := \bigcup_{y\in Y} \bigcup_{\substack{U\in\mathcal{CO}(X)\\ U\subseteq f(y)}} (U \times T_{U,y}) \in \mathcal{O}(X \times Y).$$

[If $f \in \mathcal{CO}(X)^Y$ and $T_{U,y} = f^{-1}\Big(f(y)\Big)$ $\Big(U \in \mathcal{CO}(X)$, $y \in Y$ such that $U \subseteq f(y)\Big)$, this set equals $\bigcup_{y\in Y}\Big(f(y) \times f^{-1}\Big(f(y)\Big)\Big)$, which may be reduced to a finite union since $\operatorname{Im} f$ is finite, so belongs to $\mathcal{CO}(X \times Y)$.] By Lemma 4 (1), for all $y \in Y$,

$$[\Psi(R)](y_0) = \bigcup \{ U \in \mathcal{CO}(X) \mid U \subseteq f(y) \text{ for some } y \in Y \text{ and } y_0 \in T_{U,y} \}$$

which equals $f(y_0)$. Hence $\Psi(R) = f$, so Ψ is surjective. Therefore Ψ is an order-isomorphism. ■

Theorem 6.7. *Let $L \in \mathbf{Lat}$, $P \in \mathbf{P}$. Then* $\operatorname{Con}(L^P) \cong (\operatorname{Con} L)^{\overline{P}}_{\Sigma}$.

Proof. As $\operatorname{Comp} L \in \mathbf{DSlat}$, there exists a Stone space X such that

$$\mathcal{CO}(X) \cong \operatorname{Comp} L.$$

By Proposition 6, $(\operatorname{Comp} L)^{\overline{P}} \cong \mathcal{CO}(X \times \overline{P})$, where $X \times \overline{P}$ is a Stone space by Lemmas 1 and 3. By Theorem 5.10, $[(\operatorname{Comp} L)^{\overline{P}}]^{\sigma} \cong \operatorname{Con}(L^P)$. By Lemma 5, $\mathcal{CO}(X \times \overline{P})^{\sigma} \cong \mathcal{O}(X \times \overline{P})$. By Proposition 6, $\mathcal{O}(X \times \overline{P}) \cong \mathcal{O}(X)^{\overline{P}}_{\Sigma}$. By Lemma 5 again, $\mathcal{O}(X)^{\overline{P}}_{\Sigma} \cong [(\operatorname{Comp} L)^{\sigma}]^{\overline{P}}_{\Sigma} \cong (\operatorname{Con} L)^{\overline{P}}_{\Sigma}$. Hence $\operatorname{Con}(L^P) \cong (\operatorname{Con} L)^{\overline{P}}_{\Sigma}$. ■

From Corollary 3.7, we get the following.

Corollary 6.8. *Let $L \in \mathbf{Lat}$, $M \in \mathbf{D}$. Then*

$$\operatorname{Con}(L^{P(M)}) \cong \mathbf{Slat}\Big(\big(\operatorname{Comp} L, \vee, 0_{\operatorname{Con} L}\big), \big((M_{\mathrm{Bool}})^{\sigma}, \cap, M_{\mathrm{Bool}}\big)\Big).$$

Lemma 6.9. *Let $M \in \mathbf{D}$. Then* $(M_{\mathrm{Bool}})^{\sigma} \cong \operatorname{Con} M$.

Proof. Because $P(M_{\mathrm{Bool}})$ is trivially ordered, we have

$$(M_{\mathrm{Bool}})^{\sigma} \cong \mathcal{U}\big(P(M_{\mathrm{Bool}})\big) \cong \mathcal{O}\big(P(M_{\mathrm{Bool}})\big) \cong \mathcal{O}\big(P(M)\big) \cong \operatorname{Con} M.$$ ■

Corollary 6.10. *Let $L \in \mathbf{Lat}$, $M \in \mathbf{D}$. Then*

$$\operatorname{Con}(L^{P(M)}) \cong \mathbf{Slat}\big((\operatorname{Comp} L, \vee, 0_{\operatorname{Con} L}), (\operatorname{Con} M, \cap, 1_{\operatorname{Con} M})\big).$$

The next corollary follows from Corollary 3.7.

Corollary 6.11. *Let* $L \in \mathbf{Lat}$*,* $M \in \mathbf{D}$*. Then*

$$\operatorname{Con}(L^{P(M)}) \cong (\operatorname{Con} L)_{\Lambda}^{P(\operatorname{Con} M)}.$$

Corollary 6.12 ([9], Theorem 2.1)**.** *Let* L *be a lattice and* P *a finite poset with* n *elements. Then* $\operatorname{Con}(L^P) \cong (\operatorname{Con} L)^n$*.*

Proof. As $\overline{P}$ is a discrete space, $(\operatorname{Con} L)_{\Sigma}^{\overline{P}} = (\operatorname{Con} L)^{\overline{P}}$. The result follows from Theorem 7.

■

Corollary 6.13 ([26], Theorem)**.** *Let* L *be a finite lattice,* $M \in \mathbf{D}$*. Then* $\operatorname{Con}(L^{P(M)}) \cong (\operatorname{Con} L)^{P(\operatorname{Con} M)}$*.*

Proof. As $\operatorname{Comp} L$ is finite,

$$\begin{aligned}&\mathbf{Slat}\Big((\operatorname{Comp} L, \vee, 0_{\operatorname{Con} L}), (\operatorname{Con} M, \cap, 1_{\operatorname{Con} M})\Big)\\ &\quad = \mathbf{Slat}^{\text{fin}}\Big((\operatorname{Comp} L, \vee, 0_{\operatorname{Con} L}), (\operatorname{Con} M, \cap, 1_{\operatorname{Con} M})\Big).\end{aligned}$$

By Corollary 10, the left-hand side is isomorphic to $\operatorname{Con}(L^{P(M)})$. By Corollary 3.7, the right-hand side is isomorphic to $(\operatorname{Con} L)^{P(\operatorname{Con} M)}$.

■

7. A counterexample

In this section we show that the answer to Schmidt's question (§1) is in general negative. As stated in §1, Grätzer and Schmidt have determined exactly when it has a positive solution ([15], Theorem 3); our results were obtained independently.

Lemma 7.1. *Let* S *be a chain with 0. Let* $T \in \mathbf{D}$*. Then*

$$\mathbf{Slat}\Big((S, \vee, 0_S), (T^{\sigma}, \cap, T)\Big) = \{\, f \in (T^{\sigma})^{S^{\partial}} \mid f(0_S) = T \,\}.$$

Proof. Let $f \in \mathbf{Slat}\Big((S, \vee, 0_S), (T^{\sigma}, \cap, T)\Big)$. Assume s_1, $s_2 \in S$ and $s_1 \leq s_2$. Then $f(s_2) = f(s_1 \vee s_2) = f(s_1) \cap f(s_2)$, so that $f(s_2) \subseteq f(s_1)$.

Now assume $f \in (T^{\sigma})^{S^{\partial}}$. Let s_1, $s_2 \in S$. Without loss of generality $s_1 \leq s_2$. Hence $f(s_1 \vee s_2) = f(s_2)$. As $f(s_2) \subseteq f(s_1)$, we have $f(s_1) \cap f(s_2) = f(s_2)$. Thus $f(s_1 \vee s_2) = f(s_1) \cap f(s_2)$.

■

Corollary 7.2. *Let C be a chain. Let $S := \mathbf{1} \oplus (C^{\partial})$, $T \in \mathbf{D}$. Then:*

(1) $\mathbf{Slat}\Big((S, \vee, 0_S), (T^{\sigma}, \cap, T)\Big) \cong (T^{\sigma})^{C}$;

(2) $\mathbf{Slat}^{\mathrm{fin}}\Big((S, \vee, 0_S), (T^{\sigma}, \cap, T)\Big) \cong \{\, f \in (T^{\sigma})^{C} \mid \operatorname{Im} f \text{ finite} \,\}$.

Lemma 7.3. *The poset $\{\, f \in [\mathcal{P}(\mathbb{N})^{\sigma}]^{\mathbb{N}} \mid \operatorname{Im} f \text{ finite} \,\}$ is not a complete lattice.*

Proof. For all $n_0 \in \mathbb{N}$, define the map

$$f_{n_0}\colon \mathbb{N} \to \mathcal{P}(\mathbb{N})^{\sigma}$$

as follows. For all $n \in \mathbb{N}$,

$$f_{n_0}(n) := \begin{cases} \Big\{\emptyset, \{n_0\}\Big\} & \text{if } n \geq n_0, \\ \{\emptyset\} & \text{if } n < n_0. \end{cases}$$

Then for all $n_0 \in \mathbb{N}$,

$$f_{n_0} \in P := \{\, f \in [\mathcal{P}(\mathbb{N})^{\sigma}]^{\mathbb{N}} \mid \operatorname{Im} f \text{ finite} \,\}.$$

For $n \in \mathbb{N}$,

$$\Big(\bigvee\nolimits_{[\mathcal{P}(\mathbb{N})^{\sigma}]^{\mathbb{N}}} \{\, f_{n_0} \mid n_0 \in \mathbb{N} \,\}\Big)(n) = \mathcal{P}(\{1, \ldots, n\}).$$

Suppose for a contradiction that

$$g := \bigvee\nolimits_{P} \{\, f_{n_0} \mid n_0 \in \mathbb{N} \,\}$$

exists. Then there exists $k_0 \in \mathbb{N}$ such that $k_0 \leq n$ implies $g(k_0) = g(n)$ $(n \in \mathbb{N})$. For all $n \in \mathbb{N}$, $\mathcal{P}(\{1, \ldots, n\}) \subseteq g(n)$; if $n \geq k_0$, then $\mathcal{P}(\{1, \ldots, n\}) \subseteq g(k_0)$.

Define $h\colon \mathbb{N} \to \mathcal{P}(\mathbb{N})^{\sigma}$ as follows: for all $n \in \mathbb{N}$,

$$h(n) := \begin{cases} \mathcal{P}(\{1, \ldots, n\}) & \text{if } n \leq k_0, \\ g(k_0) & \text{if } k_0 < n. \end{cases}$$

Then $h \in P$ and for all $n_0 \in \mathbb{N}$,

$$f_{n_0} \leq h.$$

Hence $g \leq h$; but $g(k_0)$ is infinite and $h(k_0)$ is finite, a contradiction. ■

Proposition 7.4. *There exist $L \in \mathbf{Lat}$ and $M \in \mathbf{D}$ such that*

$$\operatorname{Con}(L^{P(M)}) \ncong (\operatorname{Con} L)^{P(\operatorname{Con} M)}.$$

Proof. Let $M := \mathcal{P}(\mathbb{N})$. Note that $\mathbf{1} \oplus (\mathbb{N}^\partial) = \kappa\Big(\mathbf{1} \oplus (\mathbb{N}^\partial)\Big)$. It is well-known that there exists $L \in \mathbf{Lat}$ such that $\operatorname{Con} L \cong \mathbf{1} \oplus (\mathbb{N}^\partial)$ (see, for example, [27], Theorem). Hence $\operatorname{Comp} L \cong 1 \oplus (\mathbb{N}^\partial)$.

By Lemma 6.9,

$$(\operatorname{Con} L)^{P(\operatorname{Con} M)} \cong (\operatorname{Con} L)^{P(M^\sigma)}.$$

By Corollary 3.7,

$$\begin{aligned}(\operatorname{Con} L)^{P(M^\sigma)} &\cong \mathbf{Slat}^{\text{fin}}\Big((\operatorname{Comp} L, \vee, 0_{\operatorname{Con} L}), (M^\sigma, \cap, M)\Big)\\ &\cong \mathbf{Slat}^{\text{fin}}\Big(\Big(1 \oplus (\mathbb{N}^\partial), \vee, 0\Big), \Big(\mathcal{P}(\mathbb{N})^\sigma, \cap, \mathcal{P}(\mathbb{N})\Big)\Big).\end{aligned}$$

By Corollary 2 (2) we have

$$(\operatorname{Con} L)^{P(\operatorname{Con} M)} \cong \{\, f \in [\mathcal{P}(\mathbb{N})^\sigma]^{\mathbb{N}} \mid \operatorname{Im} f \text{ finite}\,\},$$

which is not a complete lattice by Lemma 3, so cannot be isomorphic to a congruence lattice.

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